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Infinite families of connected graphs with equal spectral radius

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Abstract

We present three general mechanisms for constructing infinite families of connected graphs with equal spectral radius. The first method generates families of uniformly loaded cycles by coalescing copies of a vertex-rooted connected graph at uniformly spaced vertices of a cycle. The second method is similar to the first one, except that now the load is an edge-rooted connected graph and the operation of vertex-coalescence is changed to edge-coalescence. The third method generates families of bracelets. A bracelet resembles somehow a loaded cycle but the construction mechanism is different in spirit. Among many other results, we show that if a real number r is the spectral radius of a connected graph that is not a tree, then r is realized as spectral radius by infinitely many connected graphs.

1 Introduction

All graphs considered are undirected, loopless and without multiple edges. The spectral radius $\varrho(G)$ of a graph G is the largest eigenvalue of the adjacency matrix A_G. In other words, $\varrho(G)$ is the largest root of the characteristic polynomial $\varphi(\lambda, G) =$

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$\det(\lambda I_n - A_G)$, where n is the order of G. In what follows, we consider ϱ as a function on the set $\mathcal{C}$ of connected graphs and write

$$\Phi(t) = \{G \in \mathcal{C} : \varrho(G) = t\}.$$

The spectral radius is defined for disconnected graphs as well, but in this paper we focus on connected graphs. A nonnegative real t is called a spectral number if there exists a connected graph G such that $\varrho(G) = t$. In such a case, G is said to be a realization of t. Since the characteristic polynomial of a graph is a monic polynomial with integer coefficients, spectral numbers are either integers or irrationals. One possible way of comprehending this work is to view the variable t as a target to be attained or realized by one or several connected graphs. A difficult question of graph theory is that of determining the number

$$f(t) = \mathrm{card}[\Phi(t)]$$

of connected graphs realizing a given target t, not to mention the even more difficult problem of identifying the properties of such graphs or building as many of them as possible. The situation is clear and fully understood when $t \in [0, 2]$, cf. Table 1, but it becomes involved as soon as t is above the threshold 2. Recall that the spectral radius of the path P_n of order n is equal to $\tau_n = 2\cos(\pi/(n + 1))$. The symbol $S(n_1, n_2, n_3)$ denotes the starlike tree in which removing the vertex of degree 3 leaves the disjoint paths P_{n_1}, P_{n_2}, and P_{n_3}.

spectral number t	$\Phi(t)$	$f(t)$
0	P_1	1
1	P_2	1
τ_3	P_3	1
τ_n ($n \geq 4$ even)	P_n	1
τ_n ($n \geq 5$ odd, not $11, 17, 29$)	P_n, $S(1, 1, (n-3)/2)$	2
τ_{11}	P_{11}, $S(1, 1, 4)$, $S(1, 2, 2)$	3
τ_{17}	P_{17}, $S(1, 1, 7)$, $S(1, 2, 3)$	3
τ_{29}	P_{29}, $S(1, 1, 13)$, $S(1, 2, 4)$	3
2	Smith graphs	∞

Table 1: $f(t)$ for $t \in [0, 2]$

The spectral number 2 is the smallest one that is realized by infinitely many connected graphs. The elements of the set $\Phi(2)$ are called Smith graphs. The complete list of Smith graphs can be found in many references, see for instance [8, Theorem 1.2]. Since $f(2) = \infty$ and the Cartesian product $G \times H$ of two connected graphs obeys the rule

$$\varrho(G \times H) = \varrho(G) + \varrho(H), \tag{1}$$

it follows that $f(t) = \infty$ also for $t \in \{3, 4, 5, \ldots\}$. What happens if t is a spectral number between 2 and 3? Such a spectral number is irrational of course. Due to a density result of Shearer [5], there are infinitely many spectral numbers in the open interval $(2, 3)$. Two interesting examples are the spectral radii

$$\begin{aligned} \varrho(U_1) &\approx 2.1700864866 \\ \varrho(D_1) &\approx 2.5615528128 \end{aligned}$$

of the paw graph U_1 and the diamond graph D_1, cf. Figure 1. These non-elementary connected graphs of order 4 serve as toy examples for subsequently handling more involved cases.

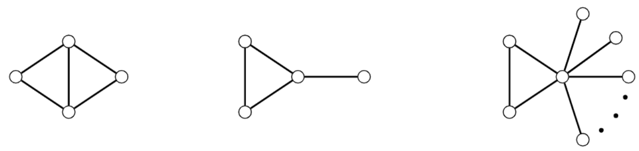

Figure 1: Diamond graph, paw graph, and paw-like graph.

It is common in practice to deal with pairs of connected graphs, not necessarily of the same order, that have the same spectral radius. In this work we construct various infinite families of connected graphs on which the spectral radius function ϱ is constant. Some of these families have an intrinsic interest and deserve a close examination. The infinite family $\Phi(2)$ of Smith graphs is formed with graphs that are well identified, but other infinite families are far more complex.

2 Getting started with two toy examples

The paw graph U_1 is a 4-vertex graph whose presence, or absence, as induced subgraph in a larger graph is a matter of importance. In chemical graph theory, U_1 represents the hydrogen depleted structure of a compound called methylcyclopropane. Since the characteristic polynomial of the paw graph can be factorized as

$$\varphi(\lambda, U_1) = (\lambda + 1)\,\underbrace{(\lambda^3 - \lambda^2 - 3\lambda + 1)}_{\phi(\lambda)},$$

we readily see that the spectral radius of U_1 is equal to

$$\pi_1 \;=\; (1/3) + (2/3)\sqrt{10}\,\cos\left((1/3)\arccos\left(10^{-3/2}\right)\right).$$

The above formula is obtained by using Viète's trigonometric representation of the largest root of the cubic polynomial ϕ. Our first result is Proposition 2.1. We need to introduce some terminology. A p-cyclic graph is a connected graph G whose cyclomatic number $c(G) = m(G) - |G| + 1$ is equal to p. Here, $|G|$ and $m(G)$ are the order and the size of G, respectively. As usual, 1-cyclic and 2-cyclic graphs are called unicyclic and bicyclic graphs, respectively. A sun graph is a unicyclic graph obtained by attaching a pendant vertex to each vertex of a cycle graph. By a broken sun we mean a sun graph that has lost at least one of its pendant vertices. Note that the paw graph is the smallest order broken sun.

Proposition 2.1. *There are infinitely many broken suns realizing the spectral radius of the paw graph.*

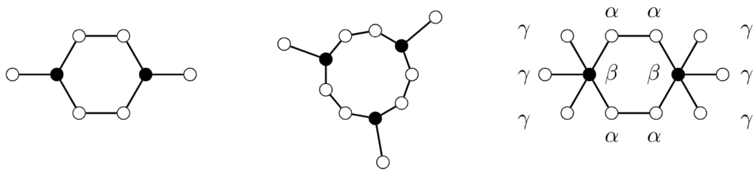

Figure 2: Left and center: U_2 and U_3. Right: Automorphism similarities in $U_{2,3}$

We accidentally came across a meaningful example of a broken sun with the same spectral radius as U_1. Such a graph, denoted U_2 and displayed in Figure 2, arises in chemical graph theory. For instance, Mukherjee and Das [4, Example 2] refer to U_2 as the *p-Xylyl* radical. The largest root of the characteristic polynomial

$$\varphi(\lambda, U_2) \;=\; (\lambda^3 + \lambda^2 - 3\lambda - 1)\,(\lambda - 1)(\lambda + 1)\,\phi(\lambda)$$

is captured by the last factor. Inspired by this example, we subsequently construct the broken sun U_3, also shown in Figure 2. With a tedious computation we obtain

$$\varphi(\lambda, U_3) \;=\; (\lambda^4 - 4\lambda^2 + \lambda + 1)^2(\lambda + 1)\,\phi(\lambda)$$

and see that the largest root is again captured by ϕ. The form of the subsequent U_k's is clear, but the computation of each $\varphi(\,\cdot\,, U_k)$ is time-consuming. So we shall prove Proposition 2.1 without relying on characteristic polynomials but by exploiting the automorphism similarity of their vertices. The notion of automorphism similarity which we use is the classical one found in Harary and Palmer [3]: two vertices u and v of a graph G are automorphically similar if $\sigma(u) = v$ for some automorphism σ of G. Automorphism similarity is an equivalence relation on the vertex set of a graph. Furthermore, the eigenvector associated to $\varrho(G)$ has a special structure according to the automorphism similarity classes. This is explained in the next lemma.

Lemma 2.1. *Let G be a connected graph with vertices $\{v_1, \dots, v_n\}$ and let $x = (x_1, \dots, x_n)^\top$ be an eigenvector of the adjacency matrix A_G associated to $\varrho(G)$. Suppose that v_i and v_j are automorphically similar, then $x_i = x_j$.*

Proof. Let $\{e_1, \dots, e_n\}$ be the canonical basis of $\mathbb{R}^n$. Let P be the permutation matrix that permutes e_i and e_j, leaving unchanged the position of the remaining canonical vectors. Since v_i and v_j are automorphically similar, $P^\top A_G P = A_G$. Hence, $P^\top A_G P x = \varrho(G)x$ or, equivalently, $A_G P x = \varrho(G) P x$. Since the eigenspace of A_G associated to $\varrho(G)$ is of dimension one, the vectors Px and x are collinear. But such vectors are positive and have the same norm. Hence, $Px = x$ and, a posteriori, $x_i = x_j$. $\square$

Our proof technique serves to prove a more general result concerning paw-like graphs. For each integer $q \geq 1$, the paw-like graph $U_{1,q}$ is obtained by attaching q pendant vertices to a given vertex of the triangle K_3, see the last graph in Figure 1. A paw-like graph is a particular instance of a circular caterpillar. The later expression refers to a unicyclic graph in which the removal of all pendant vertices results in a cycle graph.

Proposition 2.2. *Pick any integer $q \geq 1$. Then the spectral radius of $U_{1,q}$ is equal to*

$$\pi_q = \frac{1}{3} + \frac{2}{3}\sqrt{7+3q}\,\cos\left(\frac{1}{3}\arccos\left[\frac{10-9q}{(7+3q)^{3/2}}\right]\right). \tag{2}$$

Furthermore, such a spectral number is realized by infinitely many circular caterpillars.

Proof. A direct application of Proposition 1.1 in Topcu et al. [7] yields

$$\varphi(\lambda, U_{1,q}) = \lambda^{q-1}(\lambda+1)(\lambda^3 - \lambda^2 - (q+2)\lambda + q).$$

The largest root of this polynomial is found in the last factor. Viète's representation of such a root yields the characterization (2). For proving the second part of the proposition, we construct a sequence $\{U_{k,q}\}_{k\geq 1}$ of circular caterpillars growing in size, but such that

$$\varrho(U_{k,q}) = \pi_q$$

for all $k \geq 1$. The parameter q is considered as fixed. For each positive integer k, we use a cycle C_{3k} with vertices $\{v_1, \ldots, v_{3k}\}$ in clockwise order and k copies of the star $K_{1,q}$. The first copy of the star, whose leaves are denoted $\{v_{3k+1}, \ldots, v_{3k+q}\}$, is loaded on the vertex v_3 of the cycle. The second copy, whose leaves are denoted $\{v_{3k+q+1}, \ldots, v_{3k+2q}\}$, is loaded on the vertex v_6, and so on. The last copy of the star is loaded of course on the vertex v_{3k}. Loading a star on a vertex v of a cycle simply means that the central vertex of the star is glued to v. Note that $U_{k,q}$ is a circular caterpillar of order $(3+q)k$. Its vertex set is partitioned into three automorphism similarity classes: the class I contains the $2k$ vertices of degree 2, the class II contains the k vertices of degree $q+2$, and the class III contains the qk vertices of degree 1. The adjacency matrix of $U_{k,q}$ is given by

$$A_{U_{k,q}} = \begin{bmatrix} A_{C_{3k}} & M^\top \\ M & O_{qk} \end{bmatrix},$$

where $A_{C_{3k}}$ is the adjacency matrix of C_{3k}, O_d is the zero matrix of order d, and M is a block structured matrix of size $(qk)\times(3k)$. The precise form of the $\{0,1\}$-matrix M is clear from the way $U_{k,q}$ has been constructed. Now, consider the eigenvalue equation

$$A_{U_{k,q}} x = \lambda x, \tag{3}$$

where $\lambda := \varrho(U_{k,q})$ and the eigenvector x has $(3+q)k$ positive components. By taking into account the partition of the vertex set of $U_{k,q}$ into three automorphism similarity classes we see from Lemma 2.1 that x has the form

$$x = (\underbrace{(\alpha,\alpha,\beta),\ldots,(\alpha,\alpha,\beta)}_{k\ triplets},\underbrace{\gamma,\gamma,\ldots,\gamma}_{qk\ terms})^\top, \tag{4}$$

where the α-components, β-components, and γ-components are associated to the vertices of type I, II, and III, respectively. The particular case $U_{2,3}$ is illustrated in Figure 2. The vector equation (3) is a system of $(3+q)k$ equalities. By substituting (4) into (3), we obtain

$$\begin{aligned}
2k \text{ equalities of type I}: &\quad \alpha+\beta &= \lambda\alpha\\
k \text{ equalities of type II}: &\quad 2\alpha+q\,\gamma &= \lambda\beta\\
qk \text{ equalities of type III}: &\quad \beta &= \lambda\gamma.
\end{aligned}$$

We eliminate a certain number of repetitions: out of the $2k$ equalities of type I, we keep two of them; out of the k equalities of type II, we keep only one; and out of the qk equalities of type III, we keep q of them. We end up with a smaller system that can be written in the compact form

$$A_{U_{1,q}}z = \lambda z \tag{5}$$

with $z = (\alpha,\alpha,\beta,\gamma,\gamma,\ldots,\gamma)^\top$ of dimension $3+q$. Since (5) is the eigenvalue equation for the adjacency matrix of $U_{1,q}$, and z has positive components, we deduce that $U_{k,q}$ has the same spectral radius as $U_{1,q}$. □

We point out that a proof technique analogous to the one used in Proposition 2.2 has already been employed in the literature. See for instance [2, Theorem 5.5].

When $k \geq 2$, $U_{1,q}$ is smaller than $U_{k,q}$ in the sense that the first graph has fewer vertices and fewer edges than the second one. Of course, $U_{1,q}$ is neither a proper subgraph nor an induced subgraph of $U_{k,q}$, otherwise these graphs could not have the same spectral radius. Proposition 2.1 is obtained by setting $q = 1$ in Proposition 2.2. Hence, each broken sun $U_k = U_{k,1}$ realizes the spectral radius of the paw graph:

$$\{U_k : k \geq 1\} \subseteq \Phi(\pi_1).$$

We are not saying that the U_k's are the only graphs realizing π_1. Exhaustive numerical testing with connected graphs on up to 15 vertices shows that π_1 is realized by 26 graphs that are not of the type U_k, and 15 of them are not even unicyclic; cf. Table 2.

We now switch attention to our second toy example. The diamond graph D_1 is another 4-vertex graph whose presence, or absence, as induced subgraph in a larger graph is a matter of importance. The spectral radius of D_1 is equal to

$$\eta \,=\, (1/2)(1+\sqrt{17}\,).$$

This spectral number is realized by a great variety of connected graphs. Those of order 8 or less are displayed in Figure 3. Table 3 brings a bit of structure to

n	total	cyclomatic number $p=0$	$p=1$	$p\geq 2$
1,2,3	0	0	0	0
4	1	0	1	0
5,6,7	0	0	0	0
8	1	0	1	0
9	1	0	1	0
10	2	1	1	0
11	0	0	0	0
12	4	1	3	0
13	7	4	3	0
14	6	2	4	0
15	7	7	0	0

Table 2: Number of p-cyclic graphs of order n realizing the spectral radius of the paw graph.

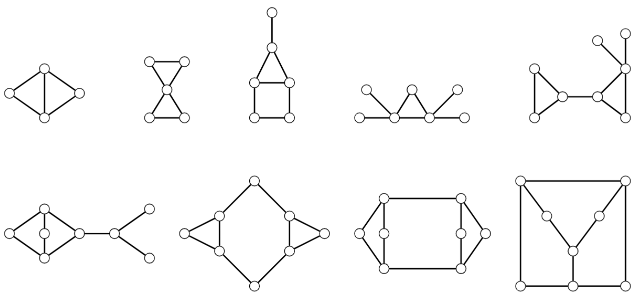

Figure 3: Connected graphs on up to 8 vertices realizing η.

this mosaic of graphs, classifying them as a function of their order and cyclomatic number. Table 3 suggests that the number of connected graphs of order n realizing η is nondecreasing as a function of n, but we do not have a formal proof of this fact. It is clear however that η is realized by a large number of connected graphs. For instance, within the connected graphs on up to 12 vertices, there are 214 realizations of η. If n is allowed to grow without bound, then we end up with $f(\eta)=\infty$. The next proposition states something stronger. Recall that the girth of a connected graph G is defined as the length of a shortest cycle contained in G as induced subgraph. The diamond graph is a bicyclic graph of girth 3.

Proposition 2.3. *For all integer $p\geq 1$, there exists a planar p-cyclic graph of girth* 3 *with the same spectral radius as the diamond graph.*

Proof. The case $p=1$ is taken care by the 4-th graph in Figure 3, so we assume that $p\geq 2$. For convenience, we introduce the change of variables $p=k+1$. For each $k\geq 1$, we construct a planar $(k+1)$-cyclic graph H_k such that

$$\varrho(H_k)=\eta. \tag{6}$$

n	total	cyclomatic number					
		$p=0$	$p=1$	$p=2$	$p=3$	$p=4$	$p\geq 5$
1,2,3	0	0	0	0	0	0	0
4	1	0	0	1	0	0	0
5	1	0	0	1	0	0	0
6	1	0	0	1	0	0	0
7	1	0	1	0	0	0	0
8	5	0	0	2	3	0	0
9	13	0	2	6	5	0	0
10	28	1	3	9	15	0	0
11	43	1	6	21	15	0	0
12	121	0	13	50	48	10	0

Table 3: Number of p-cyclic graphs of order n realizing the spectral radius of the diamond graph.

Consider a cycle C_{3k} with vertices $\{v_1, v_2, \ldots, v_{3k}\}$ in clockwise order and a group of k isolated vertices $\{v_{3k+1}, v_{3k+2}, \ldots, v_{4k}\}$. For each $i \in \{1, \ldots, k-1\}$, we connect v_{3k+i} to the consecutive vertices v_{3i} and v_{3i+1} of the cycle C_{3k}. Analogously, we connect v_{4k} to the vertices v_{3k} and v_1. The planar graph H_k constructed in this way is $(k+1)$-cyclic: it has k triangles and the main cycle C_{3k}. Since $H_1 = D_1$, equality (6) is true for $k = 1$. Suppose that $k \geq 2$. The vertex set of H_k is partitioned into three automorphism similarity classes: the class I contains the $2k$ vertices of degree 3, the class II contains the k the vertices of degree 2 on the main cycle, and the class III contains the k vertices of degree 2 off the main cycle. The adjacency matrix of H_k is given by

$$A_{H_k} = \begin{bmatrix} A_{C_{3k}} & M^\top \\ M & O_k \end{bmatrix},$$

where M is a $\{0,1\}$-matrix of size $k \times (3k)$ whose precise form is clear from the way H_k has been constructed. The rest of the proof is as in Proposition 2.2, namely, the eigenvalue equation for A_{H_k} corresponding to $\varrho(H_k)$ reduces to the eigenvalue equation for A_{H_1} corresponding to $\varrho(H_1)$. The details are omitted. □

The next result is a variant of Proposition 2.3.

Proposition 2.4. *For all integers $p \geq 1$, there exists a planar p-cyclic graph of girth 4 with the same spectral radius as the diamond graph.*

Proof. For proving the case $p = 1$, we display a concrete example: take a cycle C_4 with vertices $\{v_1, v_2, v_3, v_4\}$ in clockwise order, attach three pendant vertices to v_1 and two pendant vertices to v_3. Let $p \geq 2$. We consider again the change of variables $p = k+1$. The case $k = 1$ can be worked by hand. It suffices to compute the spectral radius of the first graph in the second row of Figure 3. Let $k \geq 2$. We construct Z_k in the same way as H_k, except that, for each $i \in \{1, \ldots, k-1\}$, the isolated vertex v_{3k+i} is connected to v_{3i} and v_{3i+2} and, analogously, v_{4k} is connected to v_{3k} and v_2.

The triangles in H_k become quadrilaterals in Z_k. Hence, Z_k is also (k+1)-cyclic but has girth 4. The rest of the proof is as before. □

It took us some time to discover the H_k's and Z_k's needed for proving Proposition 2.3 and Proposition 2.4, respectively. A few preliminary numerical tests were helpful to get some inspiration. It turns out that both sequences of graphs can be rediscovered by using the general construction mechanisms that will be developed in the next section.

3 Constructing infinite families of graphs on which ϱ is constant

The proof technique used for handling the toy examples of Section 2 is based on automorphism similarity arguments. Such a technique can be used in wider context. We next present three ways of generating families of connected graphs with a common spectral radius:

- The method UVLC consists in building Uniformly Vertex-Loaded Cycles starting from a vertex-rooted connected graph called load.
- The method UELC is a variant of the previous one. It consists in building Uniformly Edge-Loaded Cycles starting from an edge-rooted connected graph, again called load.
- The method B consists in building Bracelets starting from an edge-rooted connected graph called jewel.

We use the expression "jewel" in a broad sense (bracelets are built with all sort of jewels, after all) and not in the restricted sense of the jewel graph as defined by some authors. The methods UVLC, UELC, and B, have some features in common and it is possible to lift these techniques to a higher degree of generality. However, we prefer to give a more down-to-earth presentation with a separate discussion for each method.

3.1 Uniformly vertex-loaded cycles

The basic idea of the first method is to glue a copy of a given connected graph L, called load, at certain vertices of a cycle graph. By obvious reasons, we refer to such vertices as the supporting vertices of the cycle. Since we wish to construct a loaded cycle as symmetric as possible, we suppose that two consecutive supporting vertices are always at the same distance, say r. This explains the name of uniformly loaded cycle. The integer r can be viewed as a periodicity parameter. For carrying out a coalescence operation as mentioned above, we suppose that L is a vertex-rooted graph. To be more precise, we let $r \geq 3$ and, for each integer $k \geq 1$, we construct the graph

$$W_k = \Upsilon(L, r, k) \tag{7}$$

as follows: we take a cycle C_{kr} with vertices

$$\{v_1, \dots, v_r\} \cup \{v_{r+1}, \dots, v_{2r}\} \cup \cdots \cup \{v_{(k-1)r+1}, \dots, v_{kr}\}$$

in clockwise order and, for each $s \in \{1, \dots, k\}$, we glue a copy of L at the vertex v_{sr} of the cycle C_{kr}, i.e., we identify the root-vertex of L with the vertex v_{sr}. Such a vertex coalescence operation is carried out k times, namely, at the supporting vertices $v_r, v_{2r}, \dots, v_{kr}$. Figure 4 displays some examples: the cycles in the first row are loaded with $L = P_3$ and the periodicity parameter is $r = 4$; the cycles in the second row are loaded with $L = K_4$ and the periodicity parameter is $r = 3$.

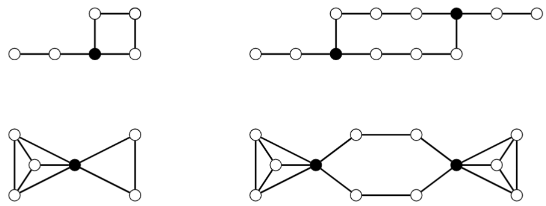

Figure 4: Examples of UVLCs. Supporting vertices are in black.

Theorem 3.1. *Let L be a vertex-rooted connected graph and $r \geq 3$ be an integer. Then all the graphs in the infinite family*

$$\mathbb{U}(L, r) = \{\Upsilon(L, r, k) : k \geq 1\}$$

have the same spectral radius.

Proof. Note that (7) is a connected graph of order $kr + k(\ell - 1)$, where ℓ is the order of L. We claim that

$$\varrho(W_k) = \varrho(W_1) \tag{8}$$

for all $k \geq 1$. Equality (8) holds tautologically for $k = 1$, so we suppose that $k \geq 2$. We denote by $u, u_1, \dots, u_{\ell-1}$ the vertices of L, with u being considered as the root-vertex. Let F be the possibly disconnected graph obtained from L by removing u and the incident edges. The adjacency matrix A_F of F is of order $\ell - 1$. Up to isomorphism, the adjacency matrix of W_1 is given by

$$A_{W_1} = \begin{bmatrix} A_{C_r} & M^\top \\ M & A_F \end{bmatrix},$$

where $M = (m_{i,j})$ is the $(\ell - 1) \times r$ matrix given by

$$m_{i,j} = \begin{cases} 1 & \text{if } j = r \text{ and } \{u, u_i\} \text{ is an edge of } L, \\ 0 & \text{otherwise.} \end{cases}$$

More generally, the adjacency matrix of W_k has the block structure

$$A_{W_k} = \left[\begin{array}{ccc|ccc} & & & M^\top & & \\ & A_{C_{kr}} & & & \ddots & \\ & & & & & M^\top \\ \hline M & & & A_F & & \\ & \ddots & & & \ddots & \\ & & M & & & A_F \end{array}\right], \tag{9}$$

where A_F shows up k times. In turn, $A_{C_{kr}}$ has the block structure

$$A_{C_{2r}} = \begin{bmatrix} A_{P_r} & E + E^\top \\ E + E^\top & A_{P_r} \end{bmatrix}, \quad A_{C_{kr}} = \begin{bmatrix} A_{P_r} & E & O & E^\top \\ E^\top & A_{P_r} & \ddots & O \\ O & \ddots & \ddots & E \\ E & O & E^\top & A_{P_r} \end{bmatrix} \quad \text{for } k \geq 3, \tag{10}$$

where A_{P_r} is the adjacency matrix of the path P_r and $E = (e_{i,j})$ is the matrix of order r given by

$$e_{i,j} = \begin{cases} 1 & \text{if } i = r, j = 1, \\ 0 & \text{otherwise}. \end{cases}$$

In the construction of W_k, besides the vertices on the cycle C_{kr}, let $u_1^{(s)}, \ldots, u_{\ell-1}^{(s)}$ be the vertices on the s-th copy of F. Hence, the vertex set of W_k can be partitioned into the following subsets:

$$\{v_1, v_{r+1}, \ldots, v_{(k-1)r+1}\}, \{v_2, v_{r+2}, \ldots, v_{(k-1)r+2}\}, \ldots, \{v_r, v_{2r}, \ldots, v_{kr}\},$$
$$\{u_1^{(1)}, \ldots, u_1^{(k)}\}, \{u_2^{(1)}, \ldots, u_2^{(k)}\}, \ldots, \{u_{\ell-1}^{(1)}, \ldots, u_{\ell-1}^{(k)}\}.$$

The vertices on each one of these subsets are automorphically similar. Hence, W_k has at most $r + \ell - 1$ automorphism similarity classes and the eigenvector x in the eigenvalue equation

$$A_{W_k} x = \lambda x, \tag{11}$$

with $\lambda := \varrho(W_k)$, can be taken as $x = (a^\top, \ldots, a^\top, b^\top, \ldots, b^\top)^\top$ with column vectors $a \in \mathbb{R}^r$ and $b \in \mathbb{R}^{\ell-1}$ appearing k times each. By substituting such an x into (11) and using (9)–(10), we obtain the system

$$\begin{cases} (A_{P_r} + E + E^\top)a + M^\top b &= \lambda a \\ Ma + A_F b &= \lambda b \end{cases} \tag{12}$$

repeated k times. We avoid unnecessary repetitions and write the above system only once. Since $r \geq 3$, we have $A_{P_r} + E + E^\top = A_{C_r}$. Hence, (12) can be written in the compact form

$$\begin{bmatrix} A_{C_r} & M^\top \\ M & A_F \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \end{bmatrix},$$

which is precisely the eigenvalue equation for A_{W_1} corresponding to the spectral radius of W_1. This shows that $\varrho(W_k) = \varrho(W_1)$ as claimed. □

Remark 3.1. For the sake of simplicity in the presentation of the theorem, we assume that $r \geq 3$. It is possible to use a periodicity parameter $r \in \{1, 2\}$, but this requires fixing a technical detail. The condition $kr \geq 3$ is necessary for making sure that C_{kr} is indeed a cycle. For this reason, we let k start from 3 when $r = 1$ and from 2 when $r = 2$. The proof of the theorem can be adapted to cover these special cases. To avoid confusion, we continue working under the assumption $r \geq 3$.

The proof of Proposition 2.2 consists essentially in applying Theorem 3.1 with $L = K_{1,q}$ and $r = 3$. It is clear that $U_{k,q} = \Upsilon(K_{1,q}, 3, k)$ is a circular caterpillar. Theorem 3.1 is applicable to a great variety of cases, because we have absolute freedom for choosing the load L and the periodicity parameter $r \geq 3$.

Example 3.1. If we pick $L = P_3$ and $r = 4$, then we get an infinite family

$$\mathbb{U}(P_3, 4) = \{\Upsilon(P_3, 4, k) : k \geq 1\}$$

of unicyclic graphs, all of them with spectral radius equal to $\varrho(\Upsilon(P_3, 4, 1)) = \left[3 + \sqrt{3}\,\right]^{1/2}$. The graph $\Upsilon(P_3, 4, 1) = P_3 * C_4$ is the first one shown in Figure 4.

3.2 Bracelets

As an alternative to UVLCs, we may consider bracelets. Instead of a load L as in Section 3.1, we now consider a "jewel" J and build a bracelet with several copies of that jewel. We get in this way an infinite family

$$\mathbb{B}(J, e) = \{B(J, e, k) : k \geq 1\}$$

of bracelets, where k represents the number of copies of the jewel and e is one of its noncut edges. The choice of e as distinguished edge and the precise construction of $B(J, e, k)$ is as follows. Since J is supposed to have at least one noncut edge, the connected graph J is not a tree. In such a case, there exists an integer $r \geq 3$ such that C_r is an induced subgraph of J. Let $\{v_1, \ldots, v_r\}$ be the vertices of C_r in clockwise order. As distinguished edge of J, we consider

$$e = \{w, v\} \quad \text{with} \quad w = v_r \quad \text{and} \quad v = v_1.$$

The choice of the cycle C_r (and its length) is irrelevant. What actually matters is the choice of e. Note that e is indeed a noncut edge of J. Let $J - e$ be the proper subgraph of J obtained by removing the edge e. Clearly, $J - e$ contains, as induced subgraph, the path P_r with vertices $\{v_1, \ldots, v_r\}$. We take k copies of $J - e$ and, for each $s \in \{1, \ldots, k\}$, we denote by $\{v_1^{(s)}, \ldots, v_r^{(s)}\}$ the vertices of the path P_r on the s-th copy of $G - e$. We connect the copies of $G - e$ by adding the edges $\{v_r^{(s)}, v_1^{(s+1)}\}$ for all $s = 1, \ldots, k - 1$ and the edge $\{v_r^{(k)}, v_1^{(1)}\}$. The bracelet $B(G, e, k)$ constructed in this way is clearly a connected graph. A bracelet resembles (and even may coincide with) a uniformly loaded circle, but in general it is not quite the same type of graph. The jewel J used to built a bracelet is not to be viewed as a load to be placed on some vertices of a circle. The building of a bracelet is a more subtle operation.

Theorem 3.2. *Let J be a connected graph other than a tree. Let e be a noncut edge of J. Then, for all integers $k \geq 1$, the bracelet $B(J, e, k)$ is a connected graph of order $k|J|$ and has the same spectral radius as J.*

Proof. Let $n = |J|$. We construct $B(J, e, k)$ as explained above. The theorem is true for $k = 1$, because $B(J, e, 1)$ coincides with J. Let $k \geq 2$. For each $s \in \{1, \dots, k\}$, let $\{u_1^{(s)}, \dots, u_{n-r}^{(s)}\}$ be the vertices of the s-th copy of $J - e$ that are not on P_r. Hence, the kn vertices of $B(J, e, k)$ are

$$v_1^{(s)}, \dots, v_r^{(s)},\ u_1^{(s)}, \dots, u_{n-r}^{(s)} \qquad \text{with } s = 1, \dots, k.$$

Let F be the possibly disconnected graph obtained from J by removing $\{v_1, \dots, v_r\}$ and their incident edges. Then, the adjacency matrix of $B(J, e, k)$ is as on the right-hand side of (9) with $A_{C_{kr}}$ as in (10). The only difference is that now $M = (m_{i,j})$ is an $(n - r) \times r$ matrix given by

$$m_{i,j} = \begin{cases} 1 & \text{if } \{u_i, v_j\} \text{ is an edge of } J, \\ 0 & \text{otherwise.} \end{cases}$$

For computing the spectral radius of $B(J, e, k)$, we observe that vertex set of $B(J, e, k)$ can be partitioned with the following subsets:

$$\begin{array}{l} \{v_1^{(1)}, \dots, v_1^{(k)}\},\ \dots, \{v_r^{(1)}, \dots, v_r^{(k)}\}, \\ \{u_1^{(1)}, \dots, u_1^{(k)}\},\ \dots, \{u_{n-r}^{(1)}, \dots, u_{n-r}^{(k)}\}. \end{array}$$

The vertices on each one of these subsets are automorphically similar. The remaining part of the proof follows the same steps as in Theorem 3.1. The details are omitted for avoiding repetitions. The conclusion is that each $B(J, e, k)$ has the same spectral radius as $B(J, e, 1) = J$. □

The next example illustrates how Theorem 3.2 works in practice.

Example 3.2. Take J as the house graph. As is well known, the house graph is of order 5 and its spectral radius is equal to the largest root of the cubic equation $\lambda^3 - 2\lambda^2 - 2\lambda + 2 = 0$, i.e.,

$$\varrho(J) \approx 2.4811943041.$$

As distinguished edge $e = \{w, v\}$, we take for instance the edge joining the black-colored vertices w and v; see the first graph in Figure 5. For each $k \geq 1$, the bracelet $B(J, e, k)$ is of order $5k$ and has the same spectral radius as J. The choice of e is not unique. We can also choose the edge on the bottom of the house. The corresponding bracelet is different from the previous one, but the spectral radius is still the same. A third possibility is to choose an edge on the roof of the house and a last possibility is to choose an edge on one side of the house. Regardless of the choice of distinguished edge and regardless of the choice of k, we always end up with a connected graph that has the same spectral radius as the house graph.

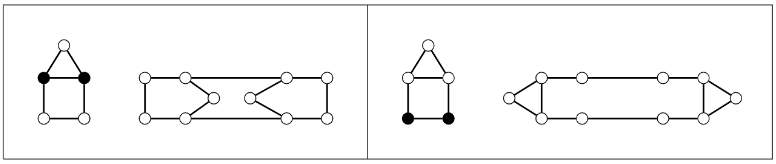

Figure 5: Jewel J (house graph) and bracelet $B(J, e, 2)$ as function of e

The next corollary is a somewhat astonishing conclusion that can be drawn from Theorem 3.2. In essence, Corollary 3.1 says that the case $f(t) = \infty$ is rather the rule and not an exception.

Corollary 3.1. *Let t be a spectral number realized by some connected graph that is not a tree. Then t is realized by infinitely many connected graphs.*

3.3 Uniformly edge-loaded cycles

Finally, we present a variant of the method UVLC. The method UELC mimics the procedure used for constructing $\Upsilon(L, r, k)$. Instead of gluing a copy of a vertex-rooted graph L at the vertices $\{v_r, v_{2r}, \ldots, v_{kr}\}$ of C_{kr}, we now suppose that the load L is an edge-rooted graph and we glue a copy of this graph at the edges

$$e_r = \{v_r, v_{r+1}\},\ e_{2r} = \{v_{2r}, v_{2r+1}\},\ \ldots,\ e_{kr} = \{v_{kr}, v_1\}$$

of C_{kr}. Clearly, $e_r = \{v_r, v_1\}$ if $k = 1$. So, we perform k edge-coalescence operations in all. By obvious reasons, $\{e_r, e_{2r}, \ldots, e_{kr}\}$ are called supporting edges of the cycle. The integer $r \geq 3$ again plays the role of a periodicity parameter. The resulting UELC is denoted by $\Gamma(L, r, k)$. For simplicity, the distinguished edge of L is not included in this notation. See Figure 6 for an example with the paw graph U_1 as load and $r = 3$ as periodicity parameter. As root-edge of the paw graph we consider the edge between the vertices of degree two.

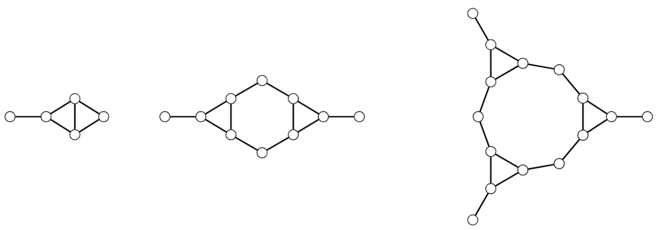

Figure 6: Uniform edge-loading of paw graphs on growing cycles

Theorem 3.3. *Let L be an edge-rooted connected graph and $r \geq 3$ be an integer. Then all the graphs in the infinite family*

$$\mathbb{E}(L, r) = \{\Gamma(L, r, k) : k \geq 1\}$$

have the same spectral radius.

Proof. This result is obtained by applying Theorem 3.2 with $J = \Gamma(L, r, 1)$. This jewel J is formed by carrying out a coalescence between the distinguished edge of L and any edge of C_r, say $\tilde{e} = \{v_r, v_1\}$. Note that J contains a cycle as induced subgraph, namely C_r. Hence, J is not a tree and admits a noncut edge. As distinguished noncut edge e of J, we choose any edge on the cycle C_r, except $\tilde{e}$. In such a case, $\Gamma(L, r, k) = B(J, e, k)$ for all $k \geq 1$, and Theorem 3.2 finishes the job. □

4 Realizing some special spectral numbers

We now apply the theorems stated in Section 3 to a large variety of situations. In particular, we construct infinite families of connected graphs realizing a given number of the form

$$\zeta(a, b) = a + \sqrt{b}\,, \tag{13}$$

where $a \geq 1$ and $b \geq 2$ are integers. For economy of language, a number as above is called a zeta number. Let us consider first the pure square root case $\sqrt{b}$, which is realized for instance by the star $K_{1,b}$. Are there other connected graphs realizing $\sqrt{b}$? Table 1 settles this question when $b \in \{1, 2, 3, 4\}$. The choice $b = 5$ is the first one for which $\sqrt{b}$ is above 2.

Proposition 4.1. *Let $b \geq 5$ be an integer. Then $\sqrt{b}$ is realized by infinitely many planar connected graphs.*

Proof. The case $b = 5$ is somewhat special. We apply Theorem 3.1 with $L = P_2$ and $r = 2$. The periodicity parameter being smaller than 3, we start with $k = 2$ as indicated in Remark 3.1. Each $\Upsilon(P_2, 2, k)$ is a broken sun realizing $\sqrt{5}$. We consider now the case $b = 2q$ with $q \geq 3$. The complete bipartite graph $K_{2,q}$ is not a tree. It is a planar connected graph and its spectral radius is equal to $\sqrt{b}$. The edges of $K_{2,q}$ are all automorphically similar (in the sense of automorphism similarity between edges), so we choose any one of them and call it e. For all $k \geq 2$, the bracelet $B(K_{2,q}, e, k)$ is a planar connected graph and, thanks to Theorem 3.2, it has the same spectral radius as $K_{2,q}$. Finally, we consider the case $b = 2q+1$ with $q \geq 3$. Let $\{v_1, v_2\}$ be the maximal degree vertices of $K_{2,q}$. Let $\widehat{K}_{2,q}$ be the planar connected graph obtained from $K_{2,q}$ by attaching one pendant vertex to v_1 and another pendant vertex to v_2. A direct computation yields the characteristic polynomial

$$\varphi(\lambda, \widehat{K}_{2,q}) = \lambda^q(\lambda^2 - 1)(\lambda^2 - 2q - 1)$$

and shows that

$$\varrho(\widehat{K}_{2,q}) = \sqrt{2q+1} = \sqrt{b}\,.$$

It now suffices to apply Theorem 3.2 with $\widehat{K}_{2,q}$ as jewel. As distinguished edge e, we choose any noncut edge of the jewel. □

Summarizing, we know that $f(\sqrt{b}\,) = \infty$ for all integers $b \geq 4$. We now pass to the case of a zeta number. An obvious example of a connected graph realizing (13) is the Cartesian product of the complete graph K_{1+a} and the star $K_{1,b}$. Indeed, the product rule (1) yields

$$\varrho\,(K_{1+a} \times K_{1,b}) \;=\; \varrho\,(K_{1+a}) + \varrho\,(K_{1,b}) \;=\; a + \sqrt{b}\,.$$

In the above Cartesian product, instead of K_{1+a}, we could choose any connected graph whose spectral radius is a. A graph is said to be of Cartesian product type if it is expressible as a Cartesian product of two or more graphs, each one with two vertices at least.

Proposition 4.2. *Let $a \geq 2$ and $b \geq 2$ be integers. Then*

- *(a) There are infinitely many unicyclic graphs realizing $1 + \sqrt{b}$.*
- *(b) There are infinitely many connected graphs of Cartesian product type realizing $a + \sqrt{b}$.*

Proof. Part (a). For each $k \geq 3$, let $\Sigma(k, b)$ be the unicyclic graph of order kb obtained by coalescing a path P_b at each vertex of the cycle C_k. Metaphorically speaking, $\Sigma(k, b)$ is a sun graph with long rays of equal length. Belardo et al. [1, Theorem 4.1] show that $\varrho(\Sigma(k, b)) = 1 + \sqrt{b}$, regardless of the choice of k. Parenthetically, the quoted result in [1] can be obtained by applying Theorem 3.1 with $L = P_b$ and periodicity parameter $r = 1$; recall Remark 3.1.

Part (b). Note that $\Phi(a - 1)$ is nonempty because it contains in particular the complete graph K_a. For any $G \in \Phi(a - 1)$ and $k \geq 3$, the product rule (1) yields

$$\varrho(G \times \Sigma(k, b)) \;=\; \varrho(G) + \varrho(\Sigma(k, b)) \;=\; (a - 1) + (1 + \sqrt{b}) \;=\; a + \sqrt{b}\,.$$

It suffices then to observe that

$$\{G \times \Sigma(k, b) : G \in \Phi(a - 1),\; k \geq 3\} \;\subseteq\; \Phi(a + \sqrt{b}\,) \tag{14}$$

and that the set on the left-hand side is formed with infinitely many connected graphs of Cartesian product type. □

Some comments on Proposition 4.2 are in order. The set on the left-hand side of (14) contains infinitely many elements, but such elements are not unicyclic graphs. Our second observation is that the inclusion (14) is strict. Indeed, $\Phi(a+\sqrt{b})$ contains many graphs that are not of Cartesian product type. By way of example, the zeta number

$$\zeta(1, 2) = 1 + \sqrt{2} \tag{15}$$

is realized by a rich variety of connected graphs, amongst which are the three highly symmetric graphs shown in Figure 7. The second graph is of Cartesian product type, but the two others are not. In turn, any one of these three graphs can be used as jewel to produce sequences of bracelets realizing (15). Four sequences of this type are shown in Figure 8. For each sequence, we draw only the particular case $k = 2$, the case of a general k can be deduced by analogy.

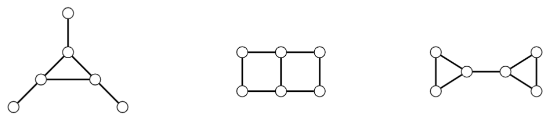

Figure 7: Connected graphs of order 6 realizing $1+\sqrt{2}$.

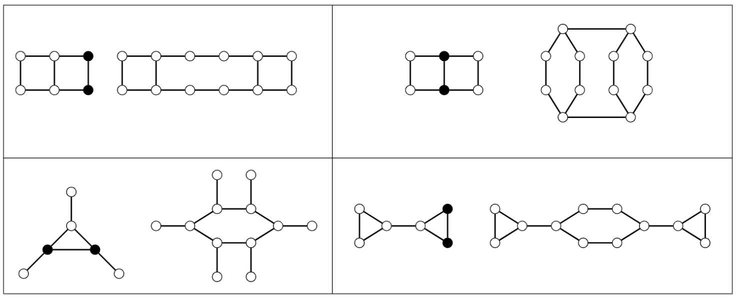

Figure 8: Jewels and corresponding bracelets realizing $1+\sqrt{2}$.

Our last result concerns the case of a spectral number of the half-zeta form

$$(1/2)\zeta(a,b) = (1/2)(a+\sqrt{b}\,).$$

There is an impressive variety of connected graphs whose spectral radius has this special form, the diamond being just the tip of the iceberg. Proposition 4.3 concerns the realization of the particular half-zeta number

$$\varrho(A_1) = (1/2)(1+\sqrt{13}\,),$$

where A_1 is the bull graph, also called the A-graph.

Proposition 4.3. *The spectral radius of the bull graph is realized by infinitely many broken suns.*

Proof. For each $k \geq 1$, we construct a graph A_k that looks similar to the broken sun U_k mentioned in Section 2. The only difference is that now we attach a pendant vertex to each vertex of C_{3k}, except to the vertices $v_3, v_6, \ldots, v_{3k}$. That A_k is a broken sun is clear (we may view A_k as a sort of complement of U_k: the missing pendant vertices in U_k show up in A_k, and vice versa). The notation A_k is consistent with the fact that A_1 is the bull graph. By exploiting the usual automorphism similarity argument, we see that $\varrho(A_k) = (1/2)(1+\sqrt{13}\,)$ for all $k \geq 1$. Alternatively, we observe that $A_k = B(A_1, e, k)$, where e is the edge of the bull graph whose removal produces the path P_5 as proper subgraph. This observation confirms that A_k has the same spectral radius as the bull graph. $\square$

Parenthetically, besides being the spectral radius of the bull graph, $(1/2)(1+\sqrt{13}\,)$ is the minimal value of the spectral radius function ϱ on the family of bicyclic graphs of order 8, cf. Simić [6, Theorem 1].

5 By way of conclusion

One can write dozens of additional propositions in the same spirit as those stated in Section 4, but there is no need of further indulging in this matter. Analyzing the two toy examples of Section 2 was an important step before arriving at the precise formulation of Theorems 3.1, 3.2, and 3.3. We could have stated these powerful theorems from the very beginning, but that is not a natural and pedagogical way of proceeding. The construction method of a bracelet is quite subtle after all and it is not the first idea that comes to mind when the aim is realizing a particular spectral number. The conceptions of the methods UVLC, UELC, and B were the final destinations after a long journey of trial-and-error. Several conclusions can be drawn from our work.

- Perhaps the most striking conclusion is this: if a nonnegative real t is realized by a connected graph that is not a tree, then it is realized by infinitely many connected graphs. We have not seen this result in the literature.

- From a practical point of view, uniformly loaded cycles and bracelets provide a large battery of examples of infinite families of connected graphs with a prescribed spectral radius. However, uniformly loaded cycles and bracelets do not cover all the possibilities. Indeed, such graphs are highly structured and have a lot of symmetry in them.

- If we focus on a specific spectral number, say the value η corresponding to the spectral radius of the diamond graph, then we quickly notice that $\Phi(\eta)$ contains a large variety of connected graphs. Table 3 and Proposition 2.3 show that if $\varrho(G) = \eta$, then the cyclomatic number $c(G)$ can be any nonnegative integer. The order, size, girth, diameter, and many other graph invariants, are also unpredictable. It is an intricate matter to identify all the members of the family $\Phi(\eta)$. The same remark applies to $\Phi(t)$ for virtually every spectral number t above 2.

There are two intriguing questions that we have not been able to solve.

Question 5.1. *Besides $t = 2$, is there another spectral number that is realized by infinitely many trees? If yes, how to identify such kind of spectral number.*

Question 5.2. *Let t be a spectral number. Table 1 shows that, when $t \in [0, 2]$, $f(t)$ can be equal to 1, 2, 3, or ∞. Are there other possibilities when $t > 2$? If yes, which is the set of values attained by the function f?*

Bracelets and uniformly loaded cycles are not trees and, therefore, such graphs do not help in answering Question 5.1. Question 5.2 is perhaps more difficult. Suppose for instance that we have been able to find four trees with the same spectral radius, say t. How can we know whether or not there exists yet another tree realizing t? Since we are considering trees of arbitrary order, numerical experimentation is of little help here.

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