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The 3-connected binary matroids with circumference 8, part I

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Abstract

This is the first paper in a sequence of three that describe the 3-connected binary matroids with circumference 8. A matroid M is said to be bent provided it has a maximum size circuit C such that M/C has a connected component with rank exceeding 1. Otherwise, it is said to be unbent. An unbent matroid M is said to be crossing when M has a maximum size circuit C, sets X and Y contained in different rank-1 connected components of M/C such that $|X| = |Y| = 2$ and $M|(C \cup X \cup Y)$ is a subdivision of $M(K_4)$. Otherwise, it is said to be uncrossing. In this paper, we construct the unbent crossing 3-connected binary matroids with circumference 8. In the second paper of this sequence, we describe the bent 3-connected binary matroids with circumference 8. In the third and final paper of this series, we deal with the unbent uncrossing 3-connected binary matroids with circumference 8.

1 Introduction

We assume familiarity with matroid theory. The notation and terminology used in this article follow Oxley [7]. For a positive integer n, we use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For a set S, the family of 2-subsets of S is denoted by $\binom{S}{2}$. We decided to start the construction of all 3-connected binary matroids having circumference 8 and large rank with the unbent crossing case because only in this case does there appear a family of 4-connected matroids.

There are many sharp extremal results in matroid theory whose bounds depend on the circumference. When one of these bounds is used to prove a theorem, it may imply that a counter-example to it must have small circumference. It is likely that the knowledge of all matroids with small circumference may simplify the proof of such a result. This was the motivation to construct the 3-connected binary matroids

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with circumference at most 7 and large rank by Cordovil, Maia Jr. and Lemos [2]. In this paper, we start to construct all 3-connected binary matroids with circumference 8 and large rank. We hope to apply our results to describe the 3-connected binary matroids with no odd circuit with size exceeding 7 extending the main result of Chun, Oxley and Wetzler [1].

Lemos and Oxley [6] establish a sharp lower bound for the circumference of a 3-connected matroid with large rank, namely:

Theorem 1.1 *Suppose that M is a 3-connected matroid. If $r(M) \geq 6$, then* $\mathrm{circ}(M) \geq 6$.

A binary matroid M is said to be a *book* having *pages* $M_1, M_2, \dots, M_n$, for $n \geq 2$, and *r-spine* T, for $r \geq 2$, provided:

(i) $M_1, M_2, \dots, M_n$ are binary matroids;

(ii) $T = E(M_1) \cap E(M_2) \cap \cdots \cap E(M_n)$;

(iii) $E(M_1) - T, E(M_2) - T, \dots, E(M_n) - T$ are pairwise disjoint sets;

(iv) $M_1|T = M_2|T = \cdots = M_n|T = K$ is isomorphic to $PG(r-1, 2)$; and

(v) $M = P_K(M_1, M_2, \dots, M_n)$, that is, the circuit space of M is spanned by $\mathcal{C}(M_1) \cup \mathcal{C}(M_2) \cup \cdots \cup \mathcal{C}(M_n)$.

The next two theorems were restated using this concept of a book proposed by Chun, Oxley and Wetzler [1].

Theorem 1.2 (Cordovil, Maia Jr. and Lemos [2]) *Let M be a 3-connected binary matroid such that $r(M) \geq 8$. Then,* $\mathrm{circ}(M) = 6$ *if and only if there is a book M' with pages $M_1, M_2, \dots, M_n$, for $n = r(M) - 2$, and 2-spine T such that, for each $i \in [n]$, M_i is isomorphic to $M(K_4)$ or F_7 and $M = M' \backslash S$, for some $S \subseteq T$.*

Let M' be as described in Theorem 1.2. Without loss of generality, we may assume that M_i is isomorphic to F_7 if and only if $i \leq m$. For $i \in [m]$, choose $a_i \in E(M_i) - T$ and set $T_i^* = E(M_i) - (T \cup a_i)$. Let $N_0 = M' \backslash \{a_1, a_2, \dots, a_m\}$. Note that N_0 is isomorphic to $M(K'''_{3,n})$. For $i \in [m]$, consider $N_i = M' \backslash \{a_j : j \in [m] \text{ and } j > i\}$. Hence $N_m = M'$. For $i \in [m]$, we have that

(1) T_i^* is a triad of N_{i-1}; and

(2) $N_{i-1} = N_i \backslash a_i$ and $T_i^* \cup a_i$ is a circuit-cocircuit of N_i.

Therefore N_i is the unique single-element binary extension of N_{i-1} obtained by adding the element a_i such that $T_i^* \cup a_i$ is a circuit. That is, M' is obtained from a matroid isomorphic to $M(K'''_{3,n})$ after a sequence of m single-element binary extensions each one adding a new element making a 4-element circuit with the elements of some triad. A similar construction can be done for M' in Theorem 1.3. This description was used to state the main results of Cordovil, Maia Jr. and Lemos [2].

Theorem 1.3 (Cordovil, Maia Jr. and Lemos [2]) *Let M be a 3-connected binary matroid such that $r(M) \geq 9$. Then,* $\mathrm{circ}(M) = 7$ *if and only if there is a book M' with pages $M_1, M_2, \ldots, M_n$, for $n = r(M) - 3$, and 2-spine T such that, for each $i \in [n-1]$, M_i is isomorphic to $M(K_4)$ or F_7, M_n is a 3-connected rank-4 binary matroid having a Hamiltonian circuit C satisfying $|T \cap C| = 2$ and $M = M' \backslash S$, for some $S \subseteq T$.*

A union of pages from a book with a 2-spine forms a 3-separating set. Consequently any matroid that appears in Theorems 1.2 or 1.3 is not internally 4-connected. The same thing happens with the main results of the next two papers of this series dealing with 3-connected binary matroids with circumference 8 (see [4, 5]). Below, we state the main result of [4] as an example. All matroids that will appear in [4, 5] are described using books with a 2-spine.

Cordovil and Lemos [3] constructed the 3-connected matroids with circumference 6. These matroids can be described using a natural generalization of a book for non-binary maroids. We do not state the result here to avoid giving this definition since it is not necessary in the remainder of this paper.

For an integer k exceeding 3, we denote by Z_k the rank-k binary spike. There is just one element of Z_k belonging to k triangles. This element is called the *tip* of Z_k. All matroids obtained from Z_k by deleting an element other than the tip are isomorphic. When $k = 4$, such a matroid is isomorphic to S_8. The *tip* of S_8 is its unique element belonging to 3 triangles.

Let M be a 3-connected binary matroid having circumference 8. We say that M is *unbent* provided, for every circuit C of M satisfying $|C| = 8$, each connected component of M/C has rank equal to 0 or 1. Otherwise, we say that M is *bent*. Now, the main result of Lemos [4] is:

Theorem 1.4 *Let M be a bent 3-connected binary matroid with circumference 8. If $r(M) \geq 14$, then there is a book M' with pages $M_1, M_2, \ldots, M_n$ and 2-spine T such that, for a fixed $e \in T$ and for each $i \in [n]$,*

(i) M_i is isomorphic to a matroid belonging to $\{Z_4, S_8, F_7, M(K_4)\}$;

(ii) when $r(M_i) = 4$, e is the tip of M_i; and

(iii) $M = M' \backslash T'$, for some $T' \subseteq T$.

Moreover, $m = |\{i \in [n] : r(M_i) = 4\}| \geq 3$ and $m + n \geq 12$.

Let M be an unbent 3-connected binary matroid having circumference 8. We say that M is *crossing* when M has an 8-element circuit C, sets X and Y contained in different rank-1 connected components of M/C such that $|X| = |Y| = 2$ and $M|(C \cup X \cup Y)$ is a subdivision of $M(K_4)$. Now, we state the main result of this paper. Its proof can be found in Section 3.

Theorem 1.5 *Let M be an unbent crossing 3-connected binary matroid with circumference 8. If $r(M) \geq 11$, then*

(i) M is a 3-connected rank-preserving restriction of M'', where M'' is a book with pages $M_1, M_2, \ldots, M_t$, for $t = r(M) - 3$, and 3-spine F such that M_i is isomorphic to $PG(3,2)$, for every $i \in [t]$; or

(ii) $M = M''\backslash T'$, where $T' \subseteq T$ and M'' is a book with pages $M_1, M_2, \ldots, M_t$, for $t = r(M) - 5$, and 2-spine T such that, for each $i \in [t] - \{1\}$, M_i is isomorphic to $K(K_4)$ or F_7 and M_1 is a 3-connected binary matroid satisfying:

(A) M_1 has a circuit D such that $|D| = 6$ and $|D \cap T| = 2$; and

(B) the simplification of M_1/T is isomorphic to F_7^* or $AG(3,2)$.

If M'' is the book described in Theorem 1.5(i), then M'' is internally 4-connected and $M''\backslash F$ is 4-connected. Both M'' and $M''\backslash F$ have circumference equal to 8. Note that $M''\backslash F$ has a rank-preserving restriction isomorphic to $M(K_{4,t})$.

Every matroid described in the conclusion of Theorem 1.4 is a bent 3-connected binary matroid with circumference 8. To restrict the matroids described in Theorem 1.5(i) so that they are contained in the class of unbent crossing 3-connected binary matroids with circumference 8 would produce a cumbersome statement (in the next paragraph, we state the condition). By Lemma 2.7(v), any matroid described in Theorem 1.5(i) has circumference at most 8. We give just one example to stress the complications that may occur. For the book M'' described in Theorem 1.5(i), choose a line L of $M''|F$. For each $i \in [t]$, let P_i be a plane of M_i containing L. Observe that $N = M''|(F \cup P_1 \cup P_2 \cup \cdots \cup P_t)$ is a rank preserving restriction of M''. But N is a book with pages $M''|F, M_1|P_1, M_2|P_2, \ldots, M_t|P_t$ and 2-spine L. By Lemma 2.7(v), its circumference is 6. (Each page of N is isomorphic to F_7.)

Let M'' be the book described in Theorem 1.5(i). We say that a subset X of $E(M'') - F$ *induces a crossing on* M'' when there is a 6-subset $\{i_1, i_2, i_3, i_4, i_5, i_6\}$ of $[t]$, a partition $\{X_1, X_2, X_3, X_4, X_5, X_6\}$ of X and a 6-subset $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ of F such that, for each $k \in [6]$, X_k is a 2-subset of $E(M_{i_k}) - F$ and $X_k \cup a_k$ is a triangle of M_{i_k}. A restriction $M''|Y$ of M'', for some $Y \subseteq E(M'')$, is a rank-preserving unbent crossing 3-connected binary matroid with circumference 8 if and only if there is $X \subseteq Y$ such that X induces a crossing on M'' and, for every $k \in [t]$, $|Y \cap [E(M_k) - F]| \geq 3$. In Lemma 2.9, we establish this fact.

Note that M_1 may have at most 35 elements in Theorem 1.5(ii). This occurs when each parallel class of M_1/T has 4 elements and its simplification is isomorphic to $AG(3,2)$. If M'' satisfies Theorem 1.5(ii)(A), then the circumference of M'' is at least 8. To see this, assume that $D \cap T = \{\alpha, \beta\}$ and chose triangles T_2 and T_3 of M_2 and M_3 respectively such that $T_2 \cap T = \{\alpha\}$ and $T_3 \cap T = \{\beta\}$. Observe that $D \triangle T_2 \triangle T_3$ is a circuit of M'' avoiding T having 8 elements. By Lemma 2.6, the circumference of M'' is at most 8. Therefore, when M'' satisfies Theorem 1.5(ii), the circumference of $M''\backslash T$ is 8.

The next results about the circuit space of a binary matroid M are used without reference throughout this paper:

(i) A cycle of M is an union of pairwise disjoint circuits of M.

(ii) The symmetric difference of circuits of M is a cycle of M.

(iii) The circuit space of M is spanned by the circuits of M and it has dimension equal to $r^*(M)$.

(iv) If C is a cycle of M and $X \subseteq E(M)$, then $C - X$ is a cycle of M/X.

2 Preliminary results

Let M be a matroid. For $F \subseteq E(M)$, an *F-arc* is a minimal non-empty subset A of $E(M) - F$ such that there exists a circuit C of M with $C - F = A$ and $C \cap F \neq \emptyset$. Note that A is an F-arc if and only if $A \in \mathcal{C}(M/F) - \mathcal{C}(M)$. The next result is Lemma 2.2 of Cordovil, Maia and Lemos [2].

Lemma 2.1 *Let M be a connected matroid. If $\emptyset \neq F \subseteq E(M), M|F$ is connected and $\operatorname{circ}(M/F) \geq 3$, then there is a circuit C of M/F such that C is an F-arc and $|C| \geq 3$.*

The next result is implicit in Cordovil, Maia and Lemos [2].

Lemma 2.2 *Let M be a connected matroid. Suppose that $M|F$ is connected, for $\emptyset \neq F \subsetneq E(M)$. If $|A| \leq 2$, for every F-arc A, then every connected component of M/F has rank equal to 0 or 1.*

Proof: The result follows because, by Lemma 2.1, $\operatorname{circ}(M/F) \leq 2$. □

We say that L is a *theta set* of a matroid M provided $L \subseteq E(M)$ and $M|L$ is a subdivision of $U_{1,3}$. When L_1, L_2 and L_3 are the series classes of $M|L$, $\{L_1, L_2, L_3\}$ is said to be *the canonical partition* of L in M. If $|L_1| = a, |L_2| = b$ and $|L_3| = c$, then L is said to be an *(a,b,c)-theta set* of M. The next result has a standard proof. We present it for completeness.

Lemma 2.3 *Let M be a matroid with circumference 8. If L is a theta set of M, then $|L| \leq 12$. Moreover, when $|L| \in \{11, 12\}$, L is an (a, b, c)-theta set of M, where $(a, b, c) \in \{(4, 4, 4), (4, 4, 3), (5, 3, 3)\}$.*

Proof: Let $\{L_1, L_2, L_3\}$ be the canonical partition of L. Assume that

$$|L_1| \geq |L_2| \geq |L_3|. \tag{2.1}$$

As $\mathcal{C}(M|L) = \{L_1 \cup L_2, L_1 \cup L_3, L_2 \cup L_3\}$, it follows that

$$\begin{aligned} 24 &= 3\text{circ}(M) \geq |L_1 \cup L_2| + |L_1 \cup L_3| + |L_2 \cup L_3| \\ &= 2(|L_1| + |L_2| + |L_3|) = 2|L|. \end{aligned} \tag{2.2}$$

Therefore $|L| \leq 12$. If $|L| = 12$, then equality holds in (2.2). In particular, $|L_1 \cup L_2| = |L_1 \cup L_3| = |L_2 \cup L_3| = 8$. Hence $|L_1| = |L_2| = |L_3| = 4$ and so L is a $(4, 4, 4)$-theta set. Assume that $|L| = 11$. By (2.2), there is a 2-subset $\{i, j\}$ of $[3]$ such that $|L_i \cup L_j| = 8$. By (2.1), we may assume that $\{i, j\} = \{1, 2\}$. Hence $|L_3| = |L| - |L_1 \cup L_2| = 11 - 8 = 3$. Thus $|L_1| = |L_2| = 4$ and L is a $(4, 4, 3)$-theta set of M or $|L_1| = 5$ and $|L_2| = 3$ and L is a $(5, 3, 3)$-theta set of M. □

Lemma 2.4 *If M is a matroid with circumference 8, then the following statements are equivalent:*

(i) M is unbent.

(ii) Every theta set of M has at most 10 elements.

Proof: Assume that M is bent. By definition, M has a circuit C such that $|C| = 8$ and M/C has a connected component with rank exceeding 1. By Lemma 2.2, there is a C-arc A of M such that $|A| \geq 3$. Therefore $C \cup A$ is a theta set of M having at least 11 elements.

Now, assume that M has a theta set L such that $|L| > 10$. By Lemma 2.3, L is an (a, b, c)-theta set of M, where $(a, b, c) \in \{(4, 4, 4), (4, 4, 3), (5, 3, 3)\}$. If $\{L_1, L_2, L_3\}$ is the canonical partition of L and $|L_1| \geq |L_2| \geq |L_3|$, then $C = L_1 \cup L_2$ is a circuit of M having 8 elements and, in M/C, L_3 is a circuit with at least 3 elements. If K is the connected component of M/C such that $L_3 \subseteq E(K)$, then $r(K) \geq |L_3| - 1 \geq 2$. Thus M is bent. □

Lemma 2.5 *Let N be 3-connected binary matroid having a triangle T such that the simplification of N/T is isomorphic to F_7^* or $AG(3, 2)$. If C is a circuit of N such that $C \cap T \neq \emptyset$, then*

$$|C - T| \leq 8 - 2|T \cap C|. \tag{2.3}$$

Proof: Assume that $|T \cap C| = 1$. In this case (2.3) becames $|C| - 1 \leq 6$. This is true because $\text{circ}(N) \leq r(N) + 1 = 7$. If $|T \cap C| = 2$, then $N/T = N/(T \cap C) \backslash (T - C)$. Thus $C - T$ is a circuit of N/T and so $|C - T| \leq 4$. Hence (2.3) follows. □

The next lemma will be used to establish that the circumference of any matroid described in Theorem 1.5(ii) is exactly 8.

Lemma 2.6 *Let N be a book having pages $N_1, N_2, \dots, N_m$, for $m \geq 3$, and 2-spine T such that N_i is isomorphic to F_7 or $M(K_4)$, for each $i \in [m] - \{1\}$. If N_1 is a 3-connected binary matroid such that the simplification of N_1/T is isomorphic to F_7^* or $AG(3, 2)$, then* $\text{circ}(N) \leq 8$.

Proof: Assume this result fails. If C is a circuit of N such that $|C| = \mathrm{circ}(N)$, then $|C| \geq 9$. For a positive integer n, $C = C_{i_1} \triangle C_{i_2} \triangle \cdots \triangle C_{i_n}$, where C_j is a cycle of N_j, for every $j \in J = \{i_1, i_2, \ldots, i_n\} \subseteq [m]$, where $1 \leq i_1 < i_2 < \cdots < i_n \leq m$. Choose J and these cycles such that n is minimum. If $n = 1$, then C is a circuit of N_{i_1}; a contradiction because $|C| \leq \mathrm{circ}(N_{i_1}) \leq r(N_{i_1}) + 1 \leq 7$. Thus $n \geq 2$. By the choice of n, $C_j - T \neq \emptyset$, for every $j \in J$. (If $C_j \subseteq T$, say $j = i_n$, then $C_{i_n} \in \{\emptyset, T\}$ is a cycle of $N_{j_{n-1}}$ and so $C_{i_{n-1}} \triangle C_{i_n}$ is a cycle of $N_{i_{n-1}}$. This cycle can replace $C_{i_{n-1}}$ and C_{i_n} in the symmetric difference that defines C; a contradiction to the minimality of n.) If $C_j \cap T = \emptyset$, for some $j \in J$, then there is a circuit D of N_j such that $D \subseteq C_j \subsetneq C$; a contradiction since D is a circuit of N. Hence $C_j \cap T \neq \emptyset$, for every $j \in J$. Therefore $|C_j \cap T| \in \{1, 2\}$, for each $j \in J$. For $j \in J$, we set

$$D_j = \begin{cases} C_j, & \text{when } |C_j \cap T| = 1, \\ C_j \triangle T, & \text{when } |C_j \cap T| = 2. \end{cases}$$

In particular, $|D_j \cap T| = 1$. Note that D_j is a circuit of N_j, otherwise C contains properly a circuit of N_j. Now, we show that

$$\text{if } \{j, j'\} \text{ is a 2-subset of } [n], \text{ then } D_j \cap T \neq D_{j'} \cap T. \tag{2.4}$$

If (2.4) fails, then $D_j \triangle D_{j'}$ is a cycle of N and so $C = D_j \triangle D_{j'} = (D_j - T) \cup (D_{j'} - T)$. If $j' < j$, then N_j is isomorphic to $M(K_4)$ or F_7 and so $|D_j - T| = 2$. Hence $9 \leq |C| = |D_{j'} - T| + 2$; a contradiction because $7 \leq |D_{j'} - T| = |D_{j'}| - 1 \leq \mathrm{circ}(N_{j'}) - 1 \leq r(N_{j'}) \leq 6$. Thus (2.4) holds. In particular, $n \leq |T| = 3$. Next, we establish that $i_1 = 1$. If $1 \notin J$, then

$$\begin{aligned} 9 \leq |C| &= |C \cap T| + |D_{i_1} - T| + |D_{i_2} - T| + \cdots + |D_{i_n} - T| \\ &= |C \cap T| + 2n \ \leq\ |C \cap T| + 6 \ \leq\ 8; \end{aligned}$$

a contradiction. Thus $1 \in J$.

Case 1. $n = 3$, say $J = \{1, 2, 3\}$.

By (2.4), $T = \{e_1, e_2, e_3\}$, where $\{e_j\} = D_j \cap T$ for $j \in J$. First, we prove that $\{e_2, e_3\} \cap \mathrm{cl}_{N_1}(D_1) = \emptyset$. If e_2 or e_3 belongs to $\mathrm{cl}_{N_1}(D_1)$, say e_2, then there is a circuit D of N_1 such that $e_2 \in D \subseteq (D_1 - e_1) \cup e_2$. Thus $D \triangle D_2 \subseteq (D_1 - e_1) \cup (D_2 - e_2)$ is a non-empty cycle of N properly contained in C; a contradiction. Therefore $\{e_2, e_3\} \cap \mathrm{cl}_{N_1}(D_1) = \emptyset$. Hence $|D_1| \leq r(N_1) = 6$. Observe that

$$C' = D_1 \triangle D_2 \triangle D_3 \triangle T = (D_1 - e_1) \cup (D_2 - e_3) \cup (D_3 - e_3)$$

is a cycle of N and so $C = C'$. Hence $9 \leq |C| = |D_1 - e_1| + |D_2 - e_2| + |D_3 - e_3| = |D_1 - e_1| + 4$. Hence $|D_1| = 6$. Now, $D = D_1 \triangle T$ is a circuit of N_1 because $\{e_2, e_3\} \cap \mathrm{cl}_{N_1}(D_1) = \emptyset$; a contradiction to Lemma 2.5 since $5 = |D - T| > 8 - 2|D \cap T| = 4$.

Case 2. $n = 2$, say $J = \{1, 2\}$.

As $C_1 \cap C_2 \neq \emptyset$, it follows that $\{C_1, C_2\} \in \{\{D_1 \triangle T, D_2\}, \{D_1, D_2 \triangle T\}, \{D_1 \triangle T, D_2 \triangle T\}\}$. Hence

$$C = \begin{cases} D_1 \triangle D_2 \triangle T = (D_1 - e_1) \cup (D_2 - e_2) \cup (T - \{e_1, e_2\}) & \text{or} \\ D_1 \triangle D_2 = D_1 \cup D_2. \end{cases}$$

The second possibility cannot occur and so

$$\begin{aligned} 9 \leq |C| &= |D_1 - e_1| + |D_2 - e_2| + |T - \{e_1, e_2\}| \\ &= |D_1 - e_1| + 2 + 1 \;=\; |D_1 - e_1| + 3. \end{aligned}$$

Therefore $|D_1| = 7$ and $D_1 - e_1$ is a basis for N_1. When $T = \{e_1, e_2, e_3\}$, there is a circuit C' of N_1 such that $e_3 \in C' \subsetneq (D_1 - e_1) \cup e_3$; a contradiction because C' is properly contained in C. □

Now, we establish a simple result. Item (ii) of the next lemma was used by Cordovil, Maia Jr. and Lemos [2] without proof. We added item (iv) in the next lemma because Theorem 1.5 (i) will become an immediate consequence of it.

Lemma 2.7 *Let N be a simple binary matroid. For $L \subseteq E(N)$ and $m \geq 2$, if the connected components $K_1, K_2, \dots, K_m$ of N/L satisfy $r(K_1) = r(K_2) = \cdots = r(K_m) = 1$, then*

- *(i) the circuit space of N is spanned by $\{C \in \mathcal{C}(N) : |C - L| \in \{0, 2\}\}$;*
- *(ii) $E(K_1), E(K_2), \dots, E(K_m)$ are pairwise disjoint cocircuits of N;*
- *(iii) when $N|L \cong PG(r-1, 2)$, for some $r \geq 2$, then N is a book with pages $N|[E(K_1) \cup L], N|[E(K_2) \cup L], \dots, N|[E(K_m) \cup L]$ and r-spine L. Moreover, each page of this book has rank equal to $r + 1$; and*
- *(iv) when $N|L \cong PG(r-1, 2)$, for some $r \geq 2$, then N is a rank-preserving restriction of a book with r-spine L and $r(N) - r$ pages, each isomorphic to $PG(r, 2)$; and*
- *(v) when $N|L \cong PG(r-1, 2)$, for some $r \geq 2$, then* $\operatorname{circ}(N) \leq 2r + 2$.

Proof: If B' and B'' are bases of $N|L$ and N/L respectively, then $B = B' \cup B''$ is a basis of N. As K_i is a connected component of N/L and $r(K_i) = 1$, it follows that $|B'' \cap E(K_i)| = 1$, say $B'' \cap E(K_i) = \{a_i\}$. Hence $B'' = \{a_1, a_2, \dots, a_m\}$. For each $b \in B^* = E(N) - B$, let C_b be the circuit of N such that $C_b - B = \{b\}$. The circuit space of N is spanned by $\mathcal{C} = \{C_b : b \in B^*\}$. Observe that (i) follows provided we establish that $|C_b - L| \in \{0, 2\}$. As $C_b - L$ is a cycle of N/L, it follows that $C_b - L$ is a disjoint union of circuits of N/L. In particular, $|C_b \cap E(K_i)| \in \{0, 2\}$, for every $i \in [m]$, and $|C_b \cap E(K_i)| = 2$ if and only if $b \in E(K_i)$ (and

$C_b \cap E(K_i) = \{b, a_i\}$). Therefore $C_b \subseteq L$, when $b \in L$, and $C_b - L = \{b, a_i\}$, when $b \in E(K_i)$. Hence (i) follows. Observe that $\mathrm{cl}_N(B - a_i) = E(N) - E(K_i)$ for each $i \in [m]$. Therefore $E(K_i)$ is a cocircuit of N. We have (ii). By the proof of (i), there is a natural partition $\{\mathcal{C}_0, \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_m\}$ of $\mathcal{C}$, where $\mathcal{C}_0 = \{C_b : b \in L\}$ and, for $i \in [m]$, $\mathcal{C}_i = \{C_b : b \in E(K_i) - a_i\}$. Note that, for $i \in [m]$, $\mathcal{C}_0 \cup \mathcal{C}_i$ spans the circuit space of $N_i = N|[E(K_i) \cup L]$ because $B' \cup b_i$ is a basis of N_i. Therefore $N = P_{N|L}(N_1, N_2, \dots, N_m)$ and (iii) holds. For $i \in [m]$, let N'_i be a matroid isomorphic to $PG(r, 2)$ such that $E(N_i) \subseteq E(N'_i)$ and $N_i = N'_i|E(N_i)$. Choose $N'_1, N'_2, \dots, N'_m$ such that $E(N'_1) - L, E(N'_2) - L, \dots, E(N'_m) - L$ are pairwise disjoint. Consider the book $N' = P_{N|L}(N'_1, N'_2, \dots, N'_m)$ having pages $N'_1, N'_2, \dots, N'_m$ and r-spine L. Note that $N = N'|E(N)$ and $m = r(M) - r(L) = r(M) - r$. Hence (iv) follows.

Now, we establish (v). Let C be a circuit of N such that $|C| = \mathrm{circ}(N)$. Assume that $|C| > 2r + 2 \geq 4$. By binary orthogonality and (ii), $C \cap E(K_1), C \cap E(K_2), \dots, C \cap E(K_m)$ are even sets that partition $C - L$. Therefore there is a partition $\{X_1, X_2, \dots, X_s\}$ of $C - L$ such that $|X_1| = |X_2| = \cdots = |X_s| = 2$ and, for each $i \in [s]$, there exists $j \in [m]$ such that $X_i \subseteq E(K_j)$. By (iii), for each $i \in [s]$, there is $a_i \in L$ such that $X_i \cup a_i$ is a triangle of N. If $a_{s+1}, a_{s+2}, \dots, a_t$ are the elements of $C \cap L$, then,

$$\text{for any 2-subset } \{i, j\} \text{ of } [t],\ a_i \neq a_j. \tag{2.5}$$

Assume that (2.5) fails. Suppose that $i < j$. If $j > s$, then $i \leq s$ and $X_i \cup a_j$ is a triangle of M contained in C. Hence $C = X_i \cup a_j$; a contradiction. Thus $i < j \leq s$. In this case $(X_i \cup a_i) \triangle (X_j \cup a_j) = X_i \cup X_j$ is a cycle of N contained in C. Hence $C = X_i \cup X_j$; a contradiction. Therefore (2.5) holds. Next, we show that

$$\text{any proper subset of } \{a_1, a_2, \dots, a_t\} \text{ is independent in } N|L. \tag{2.6}$$

Let C' be a circuit of $N|L$ contained in $\{a_1, a_2, \dots, a_t\}$, say $C' = \{a_{i_1}, a_{i_2}, \dots, a_{i_k}\}$, for $1 \leq i_1 < i_2 < \cdots < i_k \leq t$. If $i_k \leq s$, then we define $l = k$. If $s < i_k$, then there is $l \in [k-1]$ such that $i_l \leq s < i_{l+1}$. (If $s < i_1$, then $C' = C$; a contradiction because $\mathrm{circ}(N|L) = r + 1$.) Thus

$$C' \triangle (X_{i_1} \cup a_{i_1}) \triangle (X_{i_2} \cup a_{i_2}) \triangle \cdots \triangle (X_{i_l} \cup a_{i_l}) = X_{i_1} \cup X_{i_2} \cup \cdots \cup X_{i_l} \cup \{a_{i_{l+1}}, \dots, a_{i_k}\}$$

is a non-empty cycle of N contained in C. Thus it must be equal to C. Hence $k = t$, that is, $C' = \{a_1, a_2, \dots, a_t\}$. Therefore (2.6) holds. By (2.6), $t \leq \mathrm{circ}(N|L) = r + 1$ and $|C| \leq 2t = 2r + 2$; a contradiction and so (v) follows. □

The next result has a very simple proof. We omit it.

Lemma 2.8 *Let C be a cycle of a binary matroid N. If S is a series class of N such that $C - S$ is a non-empty independent set of N, then C is a circuit of N.*

In the next lemma, we use the same notation as used in Theorem 1.5(i).

Lemma 2.9 *For $t \geq 6$, let M'' be a book with pages $M_1, M_2, \ldots, M_t$ and 3-spine F such that M_i is isomorphic to $PG(3,2)$ for every $i \in [t]$. For $Y \subseteq E(M'')$, $M''|Y$ is a rank-preserving unbent crossing 3-connected binary matroid with circumference 8 if and only if there is $X \subseteq Y$ such that X induces a crossing on M'' and, for every $k \in [t]$, $|Y \cap [E(M_k) - F]| \geq 3$.*

Proof: By Lemma 2.7(v), the circumference of M'' is 8. First, we describe a maximum size circuit of M''. Let C be a circuit of M'' such that $|C| = 8$. For $i \in [t]$, set $X_i = [E(M_i) - F] \cap C$. Assume that $|X_1| \geq |X_2| \geq \cdots \geq |X_t|$. By binary orthogonality, $|X_i|$ is even. As $r(M_i) = 4$, it follows that $|X_i| \in \{0, 2, 4\}$. Let s be the biggest integer such that $|X_s| \neq 0$. For $i \in [s]$, set

$$F_i = \{a \in F : \text{there is a 2-subset } A \text{ of } X_i \text{ such that } A \cup a \text{ is a triangle of } M''\}.$$

Note that $|F_i| = 1$, when $|X_i| = 2$, and $|F_i| = 6$, when $|X_i| = 4$. Set $F_0 = C \cap F$. Now, we prove that $F_0, F_1, F_2, \ldots, F_s$ are pairwise disjoint. Suppose that $a \in F_i \cap F_j$ for $0 \leq i < j \leq s$. Let A be a 2-subset of X_j such that $A \cup a$ is a triangle of M''. As $A \cup a \not\subseteq C$, it follows that $a \notin C$ and so $i \geq 1$. If A' is a 2-subset of X_i such that $A' \cup a$ is a triangle of M'', then $A \cup A' = (A \cup a) \triangle (A' \cup a)$ is a cycle of M'' properly contained in C; a contradiction. Hence $F_0, F_1, F_2, \ldots, F_s$ are pairwise disjoint and so $|F_0| + |F_1| + \cdots + |F_s| \leq |F| = 7$. Next, we show that $|X_1| = 2$. If $|X_1| \neq 2$, then $|X_1| = 4$ and so $|F_1| = 6$. In this case, $s = 1$ and $|C| \leq 5$ or $s = 2$ and $|C| = 6$; a contradiction. Thus $|X_1| = 2$. For $i \in [s]$, we have that $F_i = \{a_i\}$, for some $a_i \in F$. Note that $D = F_0 \cup \{a_1, a_2, \ldots, a_s\}$ is a circuit of $M''|F$. Therefore $s \leq 4 - |F_0|$ and so $|C| = |F_0| + 2s = 8 - |F_0|$. Consequently $|F_0| = 0$ and $s = 4$. In resume, there is a partition $\{X_1, X_2, X_3, X_4\}$ of C such that $|X_1| = |X_2| = |X_3| = |X_4| = 2$, there are pairwise different elements i_1, i_2, i_3 and i_4 of $[t]$ such that $X_k \subseteq E(M_{i_k}) - F$ and, when $X_k \cup a_k$ is a triangle of M'' for $a_k \in F$, we have that $\{a_1, a_2, a_3, a_4\}$ is a circuit of $M''|F$. If X' and Y' are contained in different rank-1 connected components of $(M''|Y)/C$, $|X'| = |Y'| = 2$ and $M|(C \cup X' \cup Y')$ is a subdivision of $M(K_4)$, then we can take $X_5 = X'$ and $X_6 = Y'$ to construct the set $X = C \cup X' \cup Y' \subseteq Y$ that induces a crossing on $M''|Y$.

Suppose that $M''|Y$ is not 3-connected. Let $\{Z, W\}$ be a l-separation for $M''|Y$, where $l \in \{1, 2\}$, say $|Z \cap X| \geq 6$. As M'' does not have loops or parallel elements, it follows that $r(Z) \leq r(M'') - 1$. Now, we may assume that Z is closed in $M''|Y$. Observe that $|Z \cap X| \geq 10$ because $M''|X$ is a subdivision of $M(K_4)$ having every series class with size 2. Hence Z spans F in M'' and so

$$\mathrm{cl}_{M''}(Z) = \cup\{E(M_i) : i \in [t] \text{ and } [E(M_i) - F] \cap Z \neq \emptyset\}.$$

If $I = \{i \in [t] : [E(M_i) - F] \cap Z = \emptyset\}$, then $I \neq \emptyset$ and

$$W = \cup_{i \in I}[(Y - F) \cap E(M_i)].$$

Note that $r(M'') = r(Z) + |I|$ and $r(W) \geq 2 + |I|$. Therefore $r(Z) + r(W) \geq r(M'') + 2$. With this contradiction, we conclude that $M''|Y$ is 3-connected. □

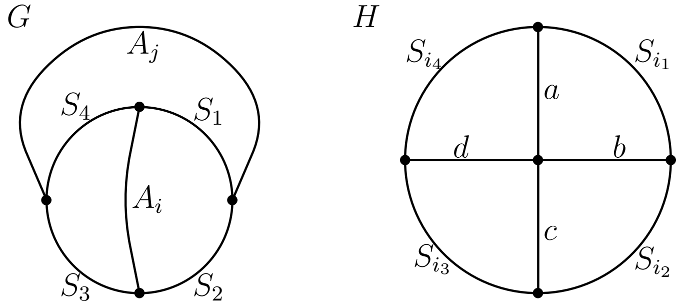

Figure 1: Graphs G and H. For $K \in \{G, H\}$, let K' be a graph obtained from K by replacing each edge uv whose label is a 2-set S by a uv-path of size 2 whose edges are labeled by the elements of S. Observe that $M(G') = M|(C \cup A_i \cup A_j)$ (see item (i) of Lemma 3.1) and that $M(H') = M|(C \cup I_l)$ in item (vi)(2) of Lemma 3.1.

3 Proof of Theorem 1.5

We first fix some of the notation used throughout this section. Let M be an unbent 3-connected binary matroid having circumference 8. Let C be a circuit of M such that $|C| = 8$. Let $H_1, H_2, \dots, H_n$ be the rank-1 connected components of M/C. By definition, when H is a connected component of M/C such that $H \notin \{H_1, H_2, \dots, H_n\}$, then $r(H) = 0$. Therefore $r(M) = 7 + n$. We assume that

$$n \geq 4 \text{ or, equivalently, } r(M) \geq 11. \tag{3.1}$$

By Lemma 2.7(ii), $E(H_1), E(H_2), \dots, E(H_n)$ are pairwise disjoint cocircuits of M.

For a 2-subset $\{i, j\}$ of $[n]$, when there are 2-subsets A_i and A_j of $E(H_i)$ and $E(H_j)$, respectively, such that $M|(C \cup A_i \cup A_j)$ is a subdivision of $M(K_4)$, we say that:

(1) A_i and A_j *cross with respect to* C; and

(2) the 2-subset $\{i, j\}$ of $[n]$ *induces a crossing on* C.

Moreover, the next two definitions are used to split the proof of Theorem 1.5 into two natural cases:

(3) M is *C-crossing* provided there is a 2-subset of $[n]$ that induces a crossing on C; and

(4) M is *strong C-crossing* provided, for every $k \in [n]$, there is a 2-subset of $[n] - \{k\}$ that induces a crossing on C.

The next lemma is the core of the proof of Theorem 1.5.

Lemma 3.1 *Let A_l be a 2-subset of $E(H_l)$, for every l belonging to the 3-subset $\{i,j,k\}$ of $[n]$. If A_i and A_j cross with respect to C, then, when C_1 and C_2 are circuits of M such that $A_i \subseteq C_1 \subseteq C \cup A_i$ and $A_j \subseteq C_2 \subseteq C \cup A_j$,*

- *(i)* $|S_1| = |S_2| = |S_3| = |S_4| = 2$, *where* $S_1 = C_1 \cap C_2, S_2 = (C_1 - C_2) \cap C, S_3 = C - (C_1 \cup C_2)$ *and* $S_4 = (C_2 - C_1) \cap C$. *(See the graph in the left in Figure 1.)*
- *(ii)* $D_1 = C_1 \triangle C_2$ *and* $D_2 = (C_1 \triangle C_2) \triangle C$ *are 8-elements circuits of M.*
- *(iii) Suppose that* $S \in \{A_k, \{e\}\}$, *where* $e \in \mathrm{cl}_M(C) - C$. *If D is a circuit of M such that* $S \subseteq D \subseteq C \cup S$, *then* $|J| \leq 2$, *where* $J = \{l \in [4] : |S_l \cap D| = 1\}$. *Moreover, if* $S = \{e\}$, *then* $|J| \leq 1$.
- *(iv) There is* $J \subseteq [4]$ *such that* $A_k \cup (\bigcup\{S_l : l \in J\})$ *is a circuit of M.*
- *(v)* $r(E(H_l)) \leq 4$, *for every* $l \in [n] - \{i,j\}$.
- *(vi) If I is an independent set of M such that $|I| = 4$ and $I \subseteq E(H_l)$, for some $l \in [n] - \{i,j\}$, then*
  - *(1) there is a 3-subset $\{a,b,c\}$ of I such that* $\{a,b\} \cup S_1 \cup S_2, \{a,c\} \cup S_1 \cup S_3\}, \{b,c\} \cup S_1 \cup S_4$ *are circuits of M and, when* $d \in I - \{a,b,c\}$, $\{a,d\} \cup S_{i_1}, \{b,d\} \cup S_{i_2}, \{c,d\} \cup S_{i_3}$ *are circuits of M, for a 3-subset $\{i_1,i_2,i_3\}$ of $[4]$; or*
  - *(2) The elements of I can be labeled by a,b,c,d such that* $\{a,b\} \cup S_{i_1}, \{b,c\} \cup S_{i_2}, \{c,d\} \cup S_{i_3}, \{d,a\} \cup S_{i_4}$ *are circuits of M, where* $[4] = \{i_1,i_2,i_3,i_4\}$. *(See the graph in the right in Figure 1.)*
- *(vii) If I is an independent set of M such that* $I \subseteq E(H_l)$ *and* $r(E(H_l)) = |I| = 3$, *for some* $l \in [n] - \{i,j\}$, *then the elements of I can be labeled by a,b,c such that*
  - *(1)* $\{a,b\} \cup S_1 \cup S_2, \{a,c\} \cup S_1 \cup S_3, \{b,c\} \cup S_1 \cup S_4$ *are circuits of M; or*
  - *(2)* $\{a,b\} \cup S_{i_1}, \{b,c\} \cup S_{i_2}, \{a,c\} \cup S_{i_1} \cup S_{i_2}$ *are circuits of M, for a 2-subset $\{i_1,i_2\}$ of $[4]$.*
- *(viii) If it is not possible to choose A_k such that A_i and A_k cross with respect to C, then* $r(E(H_k)) = 3$ *and (vii)(2) occurs for $l = k$ and an independent 3-subset I of $E(H_k)$ with* $\{i_1,i_2\} \in \{\{1,2\},\{3,4\}\}$. *Moreover, we can choose A_k such that A_j and A_k cross with respect to C.*
- *(ix) If $\{i,k\}$ does not induces a crossing on C, for every* $k \in [n] - \{i,j\}$, *then* $r(E(H_l)) = 3$, *for every* $l \in [n] - \{j\}$, *and when I_l is an independent set of M such that* $I_l \subseteq E(H_l)$, $|I_l| = 3$, *we can label the elements of I_l by a_l, b_l, c_l such that*
  - *(1)* $\{a_l,b_l\} \cup S_1, \{b_l,c_l\} \cup S_2, \{a_l,c_l\} \cup S_1 \cup S_2$ *are circuits of M; or*
  - *(2)* $\{a_l,b_l\} \cup S_3, \{b_l,c_l\} \cup S_4, \{a_l,c_l\} \cup S_3 \cup S_4$ *are circuits of M.*

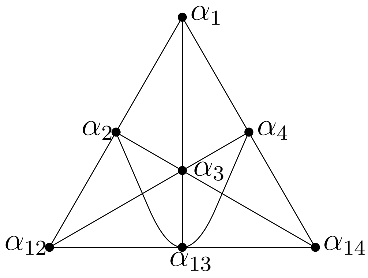

Figure 2: The geometric representation of $M'|F$ (see Lemma 3.1(xi)). It is isomorphic to F_7.

(x) *If* $r(E(H_i)) = 3$, *for* $l \in [n]$, *then* $|E(H_l)| \in \{3, 4\}$. *Moreover, when* $|E(H_l))| = 4$, $E(H_l)$ *is a circuit-cocircuit of* M.

(xi) *Let* M' *be a binary matroid such that:*

(a) $E(M) \subseteq E(M')$;

(b) $r(M) = r(M')$;

(c) $M = M'|E(M)$;

(d) $E(M') - E(M) \subseteq \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_{12}, \alpha_{13}, \alpha_{14}\} = F$;

(e) $M'|F \cong F_7$ *(see Figure 2);*

(f) $S_1 \cup \alpha_1, S_2 \cup \alpha_2, S_3 \cup \alpha_3, S_4 \cup \alpha_4$ *are triangles of* M'; *and*

(g) $|E(M')|$ *is minimum subject to the conditions (a) to (f).*

Then M' *is* 3*-connected and, when* $l \in [n]-\{i, j\}$, H_l *is a connected component of* M'/F.

(xii) *If* $A_k \cup S_l$ *is a circuit of* M, *for some* $l \in [4]$, *then* $C' = (C - S_l) \cup A_k$ *is an* 8*-element circuit of* M *such that* A_i *and* A_j *cross with respect to* C'. *Moreover, there is a rank-1 matroid* K *such that* $S_l \subseteq E(K)$ *and* K *is a connected component of both* M/C' *and* M'/F.

(xiii) *Each element of* $\text{cl}_{M'}(C)-(C\cup F)$ *is in parallel to some element of* C *in* M'/F.

Consider the circuits $C'_1 = C_1 \triangle C$ and $C'_2 = C_2 \triangle C$ of M. As $A_i \subseteq C'_1 \subseteq C \cup A_i$ and $A_j \subseteq C'_2 \subseteq C \cup A_j$, we can replace C_1 by C'_1 and/or C_2 by C'_2, when necessary, in the proof of Lemma 3.1. Observe that

$$\begin{aligned} C'_1 \cap C_2 = S_4 \quad &\text{and} \quad C - (C'_1 \cup C_2) = S_2; \\ C_1 \cap C'_2 = S_2 \quad &\text{and} \quad C - (C_1 \cup C'_2) = S_4; \\ C'_1 \cap C'_2 = S_3 \quad &\text{and} \quad C - (C'_1 \cup C'_2) = S_1. \end{aligned}$$

Depending on the choice of the circuits contained in the theta sets $A_i \cup C$ and $A_j \cup C$ to be C_1 and C_2 respectively, any series class of $M|(C \cup A_i \cup A_j)$ contained in C

can be S_1 or S_3, for example. In the proof of this lemma, when we want to prove some property of S_l, for $l \in [4]$, we just say that "by symmetry, we may assume that $l = 1$". (For l to be 1, we may need to replace C_1 by C_1' and/or C_2 by C_2' but we are not going to say that every time to avoid repetition.) The next matrix organizes the intersection of these circuits with C. The union of the sets in the first and the second lines are equal to $C_2 \cap C$ and $C_2' \cap C$ respectively. The union of the sets in the first and the second columns are equal to $C_1' \cap C$ and $C_1 \cap C$ respectively.

$$\begin{pmatrix} S_4 & S_1 \\ S_3 & S_2 \end{pmatrix}$$

Compare this matrix with the subgraph $G\backslash\{A_i, A_j\}$ of the graph G illustrated in Figure 1.

Proof: (i) First, we show that $|S_l| \geq 2$, for every $l \in [4]$. Assume that $|S_l| \leq 1$. As $M|(C \cup A_i \cup A_j)$ is a subdivision of $M(K_4)$, it follows that $S_l \neq \emptyset$. Thus $|S_l| = 1$. Observe that $L = (C \cup A_i \cup A_j) - S_l$ is a theta set of M because $r^*(M|(C \cup A_i \cup A_j)) = 3$ and S_l is a series class of $M|(C \cup A_i \cup A_j)$. Hence $|L| = |C| + |A_i| + |A_j| - |S_l| = 11$; a contradiction to Lemma 2.4. Therefore $|S_l| \geq 2$, for every $l \in [4]$. The result follows because $\{S_1, S_2, S_3, S_4\}$ is a partition of C and $|C| = 8$.

(ii) Observe that both $D_1 = A_i \cup A_j \cup S_2 \cup S_4$ and $D_2 = A_i \cup A_j \cup S_1 \cup S_3$ have 8 elements, by (i).

(iii) Replacing D by $D \triangle C$, when necessary, we may assume that $|D \cap C| \leq 4$. For $l \in J$, set $D \cap S_l = \{a_l\}$ and $S_l - D = \{b_l\}$. For clarity, we decide to divide the proof of this item into two similar parts.

Now, suppose that $|J| \in \{3, 4\}$. By symmetry, when $|J| = 3$, we may assume that $J = \{1, 2, 4\}$. Thus $D \cap S_3 = \emptyset$ because $|D \cap C| \leq 4$. Consider the following cycle of $N = M|(C \cup A_i \cup A_j \cup S)$:

$$C' = \begin{cases} D \triangle C_1 \triangle C_2 = S \cup A_i \cup A_j \cup \{a_1, b_2, a_3, b_4\}, & \text{when } |J| = 4, \\ C \triangle D \triangle C_1 \triangle C_2 = S \cup A_i \cup A_j \cup \{b_1, a_2, a_4\} \cup S_3, & \text{when } |J| = 3. \end{cases}$$

Observe that

$$C' - S = \begin{cases} A_i \cup A_j \cup \{a_1, b_2, a_3, b_4\}, & \text{when } |J| = 4, \\ A_i \cup A_j \cup \{b_1, a_2, a_4\} \cup S_3, & \text{when } |J| = 3, \end{cases}$$

is a non-empty independent set of N. By Lemma 2.8, C' is a circuit of N since S is a series class of N. We arrive at a contradiction because $|C'| \geq 9$. Thus $|J| \leq 2$.

Next, suppose that $|J| = 2$. To finish the proof of (iii), we need to establish that $S \neq \{e\}$. By symmetry, we may assume that $D \cap S_4 = \emptyset$. First, we show that $|D \cap C| = 2$. If $|D \cap C| \neq 2$, then $|D \cap C| = 4$ and $D \cap S_l \neq \emptyset$, for every $l \in [3]$. Moreover, there is a unique $l \in [3]$ such that $S_l \subseteq D$. Consider the cycle of N:

$$D' = \begin{cases} D \triangle C_1 \triangle C = S \cup A_i \cup S_1 \cup S_4 \cup \{a_2, b_3\}, & \text{when } S_1 \subseteq D, \\ D \triangle C_1 \triangle C_2 = S \cup A_i \cup A_j \cup S_4 \cup \{a_1, a_3\}, & \text{when } S_2 \subseteq D, \\ D \triangle C_2 = S \cup A_j \cup S_3 \cup S_4 \cup \{b_1, a_2\}, & \text{when } S_3 \subseteq D. \end{cases}$$

Observe that $D' - S$ is an independent set of N. By Lemma 2.8, D' is a circuit of N; a contradiction because $|D'| \geq 9$. Thus $|D \cap C| = 2$. Now, we show that $J = \{1, 3\}$. Suppose that $J \in \{\{1, 2\}, \{2, 3\}\}$. By symmetry, we may assume that $J = \{1, 2\}$. In this case, using Lemma 2.8 again, we conclude that $D \triangle C_1 \triangle C_2 = S \cup A_i \cup A_j \cup \{a_1, b_2\} \cup S_4$ is a circuit of N with at least 9 elements; a contradiction. Thus $J = \{1, 3\}$. Assume that $S = \{e\}$. In $[M|(C \cup A_i \cup A_j)]/D_1$, S_1 and S_3 are parallel classes. Hence M/D_1 has rank-1 connected components H'_1 and H'_2 such that $S_1 \subseteq E(H'_1)$ and $S_3 \subseteq E(H'_2)$ because, by (ii), D_1 is an 8-element circuit of M and M is unbent. As $D = \{e, a_1, a_3\}$ is a cycle of M/D_1, it follows that $a_1 \in X_1 = D \cap E(H'_1)$ and $a_3 \in X_2 = D \cap E(H'_2)$ are disjoint cycles of M/D_1 contained in D; a contradiction because $|X_1| \geq 2, |X_2| \geq 2$ and $|D| = 3$. Consequently $|J| \leq 1$ when $S = \{e\}$. Thus (iii) follows.

(iv) As $A_k \cup C$ is a theta-set of M, we can choose a circuit D of M such that $A_k \subseteq D \subseteq C \cup A_k$ and $|D \cap C| \leq 4$. Observe that $|D \cap C| \geq 2$, otherwise $(A_k \cup C) - (D \cap C)$ is a circuit of M with 9 elements. Assume that (iv) fails. By symmetry, we may assume that $|D \cap S_1| = 1$. Now, we show that $D \cap S_2 \neq \emptyset$. If $D \cap S_2 = \emptyset$, then $\emptyset \neq D \cap (S_3 \cup S_4) \subsetneq S_3 \cup S_4$ because $2 \leq |D \cap C| \leq 4$ and so $1 \leq |D \cap [C - (S_1 \cup S_2)]| \leq 3$. Therefore A_i and A_k cross with respect of C; a contradiction to (i) applied to A_i and A_k because $|C_1 \cap D| = 1$. Thus $D \cap S_2 \neq \emptyset$. Now, A_j and A_k cross with respect to C. By (i) applied to A_j and A_k, we have that $|D \cap (S_1 \cup S_4)| = |D \cap (S_2 \cup S_3)| = 2$. As $|D \cap S_1| = 1$, it follows that $|D \cap S_4| = 1$. Observe that A_i and A_k cross with respect to C and so $|D \cap (S_1 \cup S_2)| = |D \cap (S_3 \cup S_4)| = 2$, by (i) applied to A_i and A_k. Therefore $|D \cap S_l| = 1$, for every $l \in [4]$; a contradiction to (iii). With this contradiction, we finish the proof of item (iv).

Now, we set the notation to be used in items (v) to (vii). Assume that I is an independent set of M such that $I \subseteq E(H_l)$, for some $l \in [n]$. For a 2-subset A of I, let C_A be a circuit of M such that $A \subseteq C_A \subseteq C \cup A$. (There are two choices for C_A because $C \cup A$ is a theta set of M.) By (iv), there is a $\emptyset \neq J_A \subsetneq [4]$ such that $C_A = A \cup (\bigcup\{S_t : t \in J_A\})$. Note that

$$J_A \neq J_{A'}, \text{ when } A \text{ and } A' \text{ are different 2-subsets of } I, \tag{3.2}$$

otherwise $C_A \triangle C_{A'} = A \triangle A' \neq \emptyset$ is a cycle of M properly contained in I.

(v) Suppose that $|I| = 5$. Replacing C_A by $C \triangle C_A$, when necessary, we may assume that $4 \in J_A$. There are at most 7 different possibilities for J_A. As $\left|\binom{I}{2}\right| = 10$, it follows that I contains different 2-subsets A and A' such that $J_A = J_{A'}$; a contradiction to (3.2) and so I does not exist. Hence $r(E(H_l)) \leq 4$.

To establish items (vi) and (vii), we make a different choice for J_A. Replacing C_A by $C \triangle C_A$, when necessary, we may assume that $|J_A| \in \{1, 2\}$ and, when $|J_A| = 2$, $1 \in J_A$. Therefore

$$J_A \in \{\{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}\}. \tag{3.3}$$

(vi) Suppose that $|I| = 4$. Consider $\Upsilon = \left\{J_A : A \in \binom{I}{2}\right\}$. By (3.2), $|\Upsilon| = 6$. By (3.3), we have 7 choices for J_A. We have two cases to deal with:

Case 1. $\{\{1,2\},\{1,3\},\{1,4\}\} \subseteq \Upsilon$.

Suppose that $J_{A_1} = \{1,2\}, J_{A_2} = \{1,3\}$ and $J_{A_3} = \{1,4\}$, for 2-subsets A_1, A_2 and A_3 of I. Thus

$$C_{A_1} \triangle C_{A_2} \triangle C_{A_3} = (A_1 \triangle A_2 \triangle A_3) \cup (S_1 \cup S_2 \cup S_3 \cup S_4) = (A_1 \triangle A_2 \triangle A_3) \cup C.$$

Therefore $A_1 \triangle A_2 \triangle A_3 \subseteq I$ is a cycle of M and so $A_1 \triangle A_2 \triangle A_3 = \emptyset$. As A_1, A_2, A_3 are 2-subsets of I, it follows that $A_1 = \{a,b\}, A_2 = \{a,c\}, A_3 = \{b,c\}$, for a 3-subset $\{a,b,c\}$ of I. Thus $\{a,b\} \cup S_1 \cup S_2, \{a,c\} \cup S_1 \cup S_3, \{b,c\} \cup S_1 \cup S_4$ are circuits of M. If $d \in A$, for a 2-subset A of I and $\{d\} = I - \{a,b,c\}$, then, by (3.3), $|J_A| = 1$. By (3.2), we have (vi)(1) holds in this case.

Case 2. $\{\{1\},\{2\},\{3\},\{4\}\} \subseteq \Upsilon$.

Let G be a complete graph having I as vertex set. If $\{a,b\}$ is a 2-subset of I, then we color the edge ab of G with the color $|J_{\{a,b\}}| \in \{1,2\}$. By (3.2) and (3.3), G has 2 edges with color 2 and 4 edges with color 1. Now, we show that G has no monochromatic triangle. Assume that $\{a,b,c\}$ is a monochromatic triangle of G. The color of its edges must be 1. There is a 3-subset $\{i_1,i_2,i_3\}$ of I such that $C_{\{a,b\}} = \{a,b\} \cup S_{i_1}, C_{\{a,c\}} = \{a,c\} \cup S_{i_2}$, and $C_{\{b,c\}} = \{b,c\} \cup S_{i_3}$. Therefore $C_{\{a,b\}} \triangle C_{\{a,c\}} \triangle C_{\{b,c\}} = S_{i_1} \cup S_{i_2} \cup S_{i_3}$ is a cycle of M properly contained in C; a contradiction. Thus G has no monochromatic triangle. Hence the edges of color 2 is a perfect matching of G, say ac and bd. Therefore $C_{\{a,b\}} = \{a,b\} \cup S_{i_1}, C_{\{b,c\}} = \{b,c\} \cup S_{i_2}, C_{\{c,d\}} = \{c,d\} \cup S_{i_3}, C_{\{d,a\}} = \{d,a\} \cup S_{i_4}$, where $\{i_1,i_2,i_3,i_4\} = [I]$. We have (vi)(2). Note that $M|(C \cup I)$ is a subdivision of $M(W_4)$.

(vii) If (1) does not hold, then, by (3.2) and (3.3), there is a 2-subset $\{a,b\}$ of I such that $C_{\{a,b\}} = \{a,b\} \cup S_{i_1}$, for some $i_1 \in [4]$. We have (2) unless $C_{\{a,c\}} = \{a,c\} \cup S_1 \cup S_{j_1}$ and $C_{\{b,c\}} = \{b,c\} \cup S_1 \cup S_{j_2}$, for a 2-subset $\{j_1,j_2\}$ of [4]. Assume this is the case. Thus the cycle of M

$$\emptyset \neq C_{\{a,b\}} \triangle C_{\{a,c\}} \triangle C_{\{b,c\}} = S_{i_1} \triangle (S_{j_1} \cup S_{j_2})$$

is properly contained in C; a contradiction.

(viii) Suppose this result fails. First, we show that $r(E(H_k)) = 3$. Assume that $r(E(H_k)) > 3$. By (v), $r(E(H_k)) = 4$. If (vi)(1) happens for $l = k$, then A_i and $A_k = \{b,c\}$ cross with respect to C; a contradiction. If (vi)(2) happens for $l = k$, then A_i and A_k cross with respect to C, for some $A_k \in \{\{a,c\},\{b,d\}\}$; a contradiction. Hence $r(E(H_k)) = 3$. Now, we show that (vii)(1) cannot happen for $l = k$. If (vii)(1) occurs for $l = k$, then A_i and $A_k = \{b,c\}$ cross with respect of C; a contradiction. Thus (vii)(2) happens for $l = k$. As A_i and $A_k = \{a,c\}$ do not cross with respect to C, it follows that $\{i_1,i_2\} \in \{\{1,2\},\{3,4\}\}$. Note that A_j and $A_k = \{a,c\}$ cross with respect of C.

(ix) By (viii), (ix) follows for $l \in [n] - \{i,j\}$. We need to establish it for $l = i$. By (viii), there is a 2-subset $\{j_1,j_2\} \in \{\{1,2\},\{3,4\}\}$ such that $A_k \cup S_{j_1} \cup S_{j_2}$ is a circuit of M, for some A_k. Hence $A_k \cup S_1 \cup S_2$ is a circuit of M. Thus A_j and A_k cross. By

(viii) applied to k, j, i in place of i, j, k, we conclude that (vii)(2) holds for $l = i$ with $\{i_1, i_2\} \in \{\{1, 2\}, \{3, 4\}\}$. Hence (ix) follows also for $l = i$.

(x) If $d \in E(H_l) - I$, where I is a 3-subset of $E(H_l)$, then, by binary orthogonality, $d \cup I$ is a circuit of M. Thus d is unique and $|E(H_l)| = 4$.

(xi) As F may contain many elements of M, it follows, by the minimality of M', that M' is simple and so M' is 3-connected. By (iv), $E(H_l)$ is contained in a parallel class of M'/F. As $E(H_l)$ is a cocircuit of M, by Lemma 2.7, and $E(M) - E(H_l)$ spans F in M', it follows that $E(H_l)$ is a cocircuit of M'. Thus H_l is a connected component of M'/F.

(xii) By symmetry, we may assume that $l = 1$. As $C \cup A_k$ is a theta set of M, it follows that $C' = C \triangle (S_1 \cup A_k)$ is an 8-element circuit of M. The simplification N of $M|(C \cup A_i \cup A_j \cup A_k)$ has a non-trivial parallel class $P = E(N) \cap (S_1 \cup A_k)$, say $P = \{a, b\}$, where $a \in S_1$ and $b \in A_k$. Thus $N\backslash b$ and $N\backslash a$ are simplifications of $M|(C \cup A_i \cup A_j)$ and $M|(C' \cup A_i \cup A_j)$ respectively. Hence A_i and A_j cross with respect of both C and C'. Note that $\{A_k, S_2, S_3, S_4\}$ is the set of non-trivial series classes of $M|(C' \cup A_i \cup A_j)$ contained in C'. As $H_1, \dots, H_{k-1}, H_{k+1}, \dots, H_n$ are the rank-1 connected components of $M/(C \cup A_k) = M/(C' \cup S_1)$, it follows that M/C' has another rank-1 connected component K such that $S_1 \subseteq K$. Observe that $A_k \cup \alpha_1 = (S_1 \cup \alpha_1) \triangle (A_k \cup S_1)$ is a triangle of M'. Therefore, when we construct the matroid M' using C' instead of C, we arrive at the same matroid (up to the labeling of the elements of F). By Lemma 3.1(xi) taking C' in the place of C, we conclude that K is also a connected component of M'/F,

(xiii) Assume that $e \in \mathrm{cl}_{M'}(C) - (C \cup F)$. Let D be a circuit of M such that $e \in D \subseteq C \cup e$. There are disjoint subsets J_1 and J_2 of $[4]$ such that $|D \cap S_l| = t \in \{1, 2\}$ if and only if $l \in J_t$. By (iii) applied to $S = \{e\}$, we have that $|J_1| \leq 1$. As S_1, S_2, S_3 and S_4 are circuits of M'/F, it follows that $D' = D - \cup\{S_l : l \in J_2\}$ is a cycle of M'/F. If $J_1 = \emptyset$, then $D' = \{e\}$ and so e is spanned by F in M'; a contradiction. Thus $|J_1| = 1$, say $J_1 = \{l\}$ and $D \cap S_l = \{a_l\}$. In this case, $D' = \{e, a_l\}$ and (xiii) follows. □

Now, we describe briefly how to establish Theorem 1.5(i). Let M' be defined as in Lemma 3.1(xi). In items (xi) and (xii) of Lemma 3.1, we give conditions for M'/F to have many rank-1 connected components. When we are lucky and all connected components of M'/F have rank equal to 1, we can apply Lemma 2.7(iv) to conclude that Theorem 1.5(i) holds. This strategy will be used three times to obtain Theorem 1.5(i). In the remaining case, we need another decomposition to get Theorem 1.5(ii).

To describe M, we need to divide the analysis into two cases with different approaches.

Case 1. It is possible to choose C such that M is strong C-crossing.

Lemma 3.2 *If $X = \mathrm{cl}_M(C) - C$, then there is a partition $\{S_1, S_2, S_3, S_4\}$ of C such that S_1, S_2, S_3, S_4 are non-trivial series classes of $M\backslash X$. Moreover, when M' is defined as in Lemma 3.1(xi), $H_1, H_2, \dots, H_n$ are connected components of M'/F.*

Proof: Let G be a simple graph having $[n]$ as vertex set such that $ij \in E(G)$ if and only if $\{i, j\}$ is a 2-subset of $[n]$ that induces a crossing on C. (Remember that $n \geq 4$, by (3.1).) By hypothesis, for each $i \in [n]$, there is an edge of G not incident to i. By Lemma 3.1(viii), for each $ij \in E(G)$ and $k \in [n] - \{i, j\}$, we have that $E(G) \cap \{ik, jk\} \neq \emptyset$. Thus G contains a matching Y such that $|Y| = 2$. After a reordering of H_i's, we may assume that $Y = \{12, 34\}$, that is,

$$\text{both } \{1, 2\} \text{ and } \{3, 4\} \text{ induce a crossing on } C. \tag{3.4}$$

For $\{i, j\} \in \{\{1, 2\}, \{3, 4\}\}$, let A_i and A_j be respectively a 2-subset of $E(H_i)$ and $E(H_j)$ such that A_i and A_j cross with respect to C. Consider $N_{ij} = M\backslash[X \cup (E(H_i) - A_i) \cup (E(H_j) - A_j)]$. By Lemmas 2.7(i) and 3.1(i)(iv), there are partitions $\{S_1, S_2, S_3, S_4\}$ and $\{S'_1, S'_2, S'_3, S'_4\}$ of C such that S_1, S_2, S_3, S_4 are non-trivial series classes of N_{12} and S'_1, S'_2, S'_3, S'_4 are non-trivial series classes of N_{34}. As $N_{12}|(C \cup A_1 \cup A_2 \cup A_3 \cup A_4) = N_{34}|(C \cup A_1 \cup A_2 \cup A_3 \cup A_4)$, it follows that $\{S_1, S_2, S_3, S_4\} = \{S'_1, S'_2, S'_3, S'_4\}$. The result follows because the circuit space of $M\backslash X$ is spanned by $\mathcal{C}(N_{12}) \cup \mathcal{C}(N_{34})$. To conclude that Lemma 3.1(v)(vi)(vii)(xi)(xiii) holds for every $l \in [n]$ and Lemma 3.1(xii) holds for every $k \in [n]$, we apply Lemma 3.1 for an $\{i, j\} \in \{\{1, 2\}, \{3, 4\}\}$ such that $l \notin \{i, j\}$ and $k \notin \{i, j\}$ respectively. By Lemma 3.1(xi), $H_1, H_2, \dots, H_n$ are connected components of M'/F. □

Lemma 3.3 *Using the partition $\{S_1, S_2, S_3, S_4\}$ of C obtained in Lemma 3.2, if M' is the matroid described in Lemma 3.1(xi), then the connected components of M'/F are $H_1, H_2, \dots, H_n, K_1, K_2, K_3, K_4$, where $r(K_i) = 1$ and $S_i \subseteq E(K_i)$, for every $i \in [4]$.*

Proof: By Lemma 3.2, $H_1, H_2, \dots, H_n$ are connected components of M'/F. For $i \in [4]$, there is a parallel class P_i of M'/F such that $S_i \subseteq P_i$ because S_i is a circuit of M'/F. If K_i is a connected component of M'/F such that $P_i \subseteq E(K_i)$, then $E(K_i) \subseteq E(M') - [F \cup E(H_1) \cup E(H_2) \cup \dots \cup E(H_n)] = \mathrm{cl}_M(C) - F$ because $H_1, H_2, \dots, H_n$ are connected components of M'/F. By Lemma 3.1(xiii), $\mathrm{cl}_M(C) - F = P_1 \cup P_2 \cup P_3 \cup P_4$. Thus $E(K_i) \subseteq P_1 \cup P_2 \cup P_3 \cup P_4$. If $E(K_i) = P_i$, for every $i \in [4]$, then Lemma 3.3 follows. Assume that $E(K_i) \neq P_i$, for some $i \in [4]$, say $i = 1$. As $r_{M'}(C) = 7$ and $r_{M'}(F) = 3$, it follows that $r_{M'/F}(C) = 4$. For $i \in [4]$, choose $a_i \in S_i$. Hence $r_{M'/F}(\{a_1, a_2, a_3, a_4\}) = 4$. Let K be the simplification of K_1 such that $E(K) \subseteq \{a_1, a_2, a_3, a_4\}$. We arrive at a contradiction because $E(K)$ is independent in K. □

By Lemma 3.3, we can apply Lemma 2.7(iv) to M' to obtain Theorem 1.5(i), when Case 1 happens.

Case 2. It is not possible to choose C such that M is strong C-crossing. Choose C such that M is C-crossing (but M is not strong C-crossing).

By definition of strong C-crossing, there is $j \in [n]$ such that no 2-subset of $[n] - \{j\}$ induces a C-crossing. When necessary, we can reorder $H_1, H_2, \dots, H_n$ so that $j = 1$ and $\{1, 2\}$ induces a crossing on C. By Lemma 3.1(i), there is a partition $\{S_1, S_2, S_3, S_4\}$ of C and 2-subsets A_1 and A_2 of $E(H_1)$ and $E(H_2)$ respectively such that $|S_1| = |S_2| = |S_3| = |S_4| = 2$ and $C_1 = A_1 \cup S_1 \cup S_4$ and $C_2 = A_2 \cup S_1 \cup S_2$ are circuits of M. (We are applying Lemma 3.1 for $i = 2$ and $j = 1$.)

By Lemma 3.1(viii), $\{1, l\}$ induces a crossing on C, $r(E(H_l)) = 3$ and, for an independent set of $E(H_l)$, Lemma 3.1(vii)(2) occurs with $\{i_1, i_2\} \in \{\{1, 2\}, \{3, 4\}\}$, for every $l \in [n] - \{1\}$ (depending of the value of l, use $\{1, 2\}$ or $\{1, 3\}$ as the set that induces a crossing on C to apply this lemma). For an integer m satisfying $1 \leq m \leq n$, we may assume that Lemma 3.1(vii)(2) occurs with $\{i_1, i_2\} = \{1, 2\}$, for every l such that $2 \leq l \leq m$, and Lemma 3.1(vii)(2) occurs with $\{i_1, i_2\} = \{3, 4\}$, for every l such that $m + 1 \leq l \leq n$. That is, for $l \in [n] - \{1\}$, there is an independent 3-set I_l of M contained in $E(H_l)$, say $I_l = \{a_l, b_l, c_l\}$, such that:

(1) $\{a_l, b_l\} \cup S_1, \{b_l, c_l\} \cup S_2, \{a_l, c_l\} \cup S_1 \cup S_2$ are circuits of M, for every l such that $2 \leq l \leq m$; and

(2) $\{a_l, b_l\} \cup S_3, \{b_l, c_l\} \cup S_4, \{a_l, c_l\} \cup S_3 \cup S_4$ are circuits of M, for every l such that $m + 1 \leq l \leq n$.

(If $m = 1$, then (2) occurs for every $l \in [n] - \{1\}$. If $m = n$, then (1) occurs for every $l \in [n] - \{1\}$.)

When $m = 1$ or $m = n$, we say this C-crossing is *homogeneous*. When $2 \leq m < n$, we say this C-crossing is *heterogeneous*.

Subcase 2.1. We can choose C such that the C-crossing is heterogeneous.

Thus $2 \leq m < n$. Consequently the partition $\{S_1, S_2, S_3, S_4\}$ of C is defined by (1) and (2) applied to $l = 2$ and $l = n$ respectively. Therefore this partition does not depend on the choice of A_1. Let M' be the matroid defined in Lemma 3.1(xi). By Lemma 3.1(xi) applied to $l \in [n] - \{1\}$, we conclude that H_l is a connected component of M'/F. By Lemma 3.1(xii), for each $t \in [4]$, there is a rank-1 connected component K_t of M'/F such that $S_t \subseteq E(K_t)$. Therefore $H_2, H_3, \dots, H_n, K_1, K_2, K_3, K_4$ are rank-1 connected components of M'/F. As M'/F does not have loops, it follows, by (3.1), that M'/F has just another connected component that must have rank 1. This connected component must be H_1. Again, we obtain the book decomposition described Theorem 1.5(i) as an immediate consequence of Lemma 2.7(iv) applied to M'.

Subcase 2.2. We cannot choose C such that the C-crossing is heterogeneous. Choose C such that the C-crossing is homogeneous.

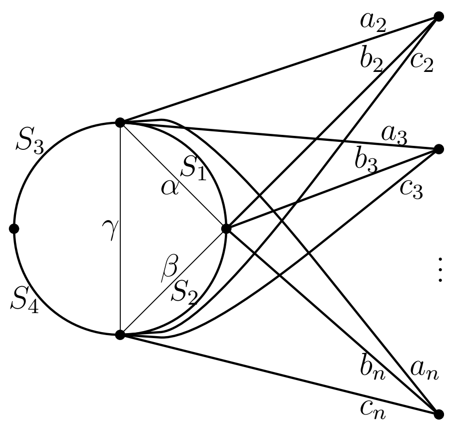

Figure 3: A graph that illustrates the possibility of $m = n$ in Subcase 2.2.

Assume that $m = 1$ or $m = n$. By symmetry, we may assume that $m = n$. In Figure 3, we illustrate this situation. The roles of α, β and γ are described in the next paragraph. Assume also that Theorem 1.5(i) does not hold. By Lemma 2.7(iv),

$$M'/F \text{ must have a connected component with rank exceeding 1.} \tag{3.5}$$

Let M'' be a matroid such that $E(M) \subseteq E(M''), r(M) = r(M''), M''|E(M) = M, E(M'') = E(M) \cup \{\alpha, \beta, \gamma\}, \alpha \cup S_1, \beta \cup S_2$ and $T = \{\alpha, \beta, \gamma\}$ are triangles of M'' and M'' is simple (that is, some of the elements of $\{\alpha, \beta, \gamma\}$ may belong to M). Hence M'' is 3-connected. By (1), for $l \in [n] - \{1\}$, $\{a_l, b_l, c_l\}$ is contained in a parallel class of M''/T and so, Lemma 3.1(x), $E(H_l)$ is contained in a parallel class of M''/T. As $E(H_l)$ is a cocircuit of M'', it follows that $H_2, H_3, \dots, H_n$ are connected components of M''/T. For $l \in [n] - \{1\}$, set $M_l = M''|(T \cup E(H_i))$. (Observe that if we rename α_1, α_2 and α_{12} in $M' \backslash [\{\alpha_3, \alpha_4, \alpha_{13}, \alpha_{1,4}\} - E(M)]$ by α, β and γ respectively, then we obtain M''.)

Now, we prove that M''/T has a rank-1 connected component K_l such that $S_l \subseteq E(K_l)$, for each $l \in [2]$, say $l = 1$. By Lemma 3.1(xii) and (1), $C' = C \triangle (\{a_n, b_n\} \cup S_1) = (C - S_1) \cup \{a_n, b_n\}$ is a circuit of M such that A_1 and A_2 cross with respect to C'. By Lemma 3.1(xii), there is a rank-1 connected component K_1 of M/C' such that $S_1 \subseteq E(K_1)$. Set $S'_1 = \{a_n, b_n\}$. By (1), for $l \in [n-1] - \{1\}$, $(\{a_l, b_l\} \cup S_1) \triangle (S'_1 \cup S_1) = \{a_l, b_l\} \cup S'_1$ is a circuit of M. Similarly $\{a_l, c_l\} \cup S'_1 \cup S_2$ is a circuit of M. Hence

(3) $\{a_l, b_l\} \cup S'_1, \{b_l, c_l\} \cup S_2, \{a_l, c_l\} \cup S'_1 \cup S_2$ are circuits of M, for every l such that $2 \leq l \leq n - 1$.

If $S_1 = \{a'_n, b'_n\}$, then $\{a'_n, b'_n\} \cup S'_1$ is a circuit of M. Choose $c'_n \in E(K_1) - \{a'_n, b'_n\}$ such that $I'_n = \{a'_n, b'_n, c'_n\}$ is independent in M. As the C'-crossing is homogeneous and (3) holds for $l \in \{2, 3\}$ since $n \geq 4$, it follows that

(4) $\{a'_n, b'_n\} \cup S'_1, \{b'_n, c'_n\} \cup S_2, \{a'_n, c'_n\} \cup S'_1 \cup S_2$ are circuits of M or

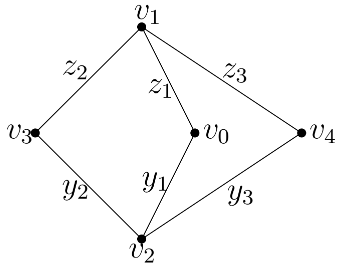

Figure 4: A graph G such that $(M''/T)|(S_3 \cup S_4 \cup A) = M(G)$.

(5) $\{b'_n, a'_n\} \cup S'_1, \{a'_n, c'_n\} \cup S_2, \{b'_n, c'_n\} \cup S'_1 \cup S_2$ are circuits of M

because $\{a'_n, b'_n\} \cup S'_1$ is a circuit of M. When we use C' instead of C to construct M'', we obtain the same matroid because M'' is defined by (1) or (3) for $l = 2$. By the previous paragraph applied to C' instead of C, we conclude that K_1 is a connected component of M''/T. For $l \in [2]$, set $M_{n+l} = M''|(T \cup E(K_l))$.

If $M_1 = M''\backslash[E(H_2) \cup E(H_3) \cup \cdots \cup E(H_n) \cup E(K_1) \cup E(K_2)]$, then M'' is a book having $M_1, M_2, \ldots, M_n, M_{n+1}, M_{n+2}$ as pages and spine T. Moreover, for each $l \in [n+2] - \{1\}$, M_l is isomorphic to $M(K_4)$ or F_7 because $r(M_l) = 3$.

To conclude the proof of Theorem 1.5(ii), we need to verify that M_1 satisfies (A) and (B).

Consider $K = M''\backslash[E(H_2) \cup E(H_3) \cup \cdots \cup E(H_n) \cup (E(K_1) - S_1) \cup (E(K_2) - S_2)]$. Observe that S_1 and S_2 are non-trivial series classes of K such that $K\backslash(S_1 \cup S_2) = M_1$. Let H be a cosimplification of K. Assume that $E(H) \cap S_1 = \{a\}$ and $E(H) \cap S_2 = \{b\}$. As $S_1 \cup \alpha$ and $S_2 \cup \beta$ are triangles of K, it follows that $\{a, \alpha\}$ and $\{b, \beta\}$ are parallel classes of H. Moreover, $H\backslash\{a, b\} = M_1$. This construction permits one to obtain a circuit of $K\backslash T$ from a circuit of M_1 by replacing α and β by respectively the elements of S_1 and S_2 (and vice-versa). In the first cases, we use symmetric differences to go from one of these circuits to the other.

Observe that $D = C \triangle (\alpha \cup S_1) \triangle (\beta \cup S_2) = \{\alpha, \beta\} \cup S_3 \cup S_4$ is a circuit of M_1. Consequently M_1 satisfies (A) of Theorem 1.5(ii).

Now, we describe the matroid M_1. Let A be a 2-subset of $E(H_1)$ such that $A \cup S_1 \cup S_4$ is a circuit of M. Thus $(A \cup S_1 \cup S_4) \triangle (S_1 \cup \alpha) = A \cup S_4 \cup \alpha$ is a circuit of M_1. Assume that $S_3 = \{y_2, z_2\}, S_4 = \{y_3, z_3\}$ and $A = \{y_1, z_1\}$. If G is the graph in Figure 4, then $(M''/T)|(S_3 \cup S_4 \cup A) = M(G)$. Observe that $\{z_1, y_1, y_2, y_3\}$ is a basis of M_1/T. For $e \in E(M_1) - (S_3 \cup S_4 \cup A \cup T)$, let C_e be a circuit of M_1/T such that $e \in C_e \subseteq e \cup S_3 \cup S_4 \cup A$. For $i \in [3]$, there is $X_i \subseteq \{y_i, z_i\}$ such that $C_e = e \cup X_1 \cup X_2 \cup X_3$. Choose C_e such that $x_e = |X_1| + |X_2| + |X_3|$ is minimum. First, we establish that

$$\text{if } \{i, j\} \text{ is a 2-subset of } [3], \text{ then } |X_i| + |X_j| \leq 2. \tag{3.6}$$

Assume that (3.6) fails. As $\{y_i, y_j, z_i, z_j\}$ is a circuit of M_1/T, it follows that $|X_i| + |X_j| = 3$, say $X_i \subseteq C_e$ and $y_j \in C_e$. Observe that $D = C_e \triangle \{y_i, y_j, z_i, z_j\} =$

$(C_e \cup x_j) - (X_i \cup y_j)$ is a cycle of M_1/T. If D' is a circuit of M_1/T such that $e \in D' \subseteq D$, then D' is contrary to the choice of C_e because $|D' \cap (A \cup S_3 \cup S_4)| \leq x_e - 2$. Thus (3.6) follows. Now, we consider the three possibilities for x_e. If $x_e = 1$, then e is in parallel with some element of $S_3 \cup S_4 \cup A$ in M_1/T. The other two possibilities for x_e are dealt in the next two lemmas.

There is a circuit C'_e of M_1 such that $C'_e - T = C_e$ and $C'_e \cap T \subseteq \{\alpha, \beta\}$. Making the symmetric difference of C'_e with the triangles $S_1 \cup \alpha$ and $S_2 \cup \beta$, when necessary, we obtain the following circuit D_e of M:

$$D_e = \begin{cases} C_e, & \text{if } C'_e \cap \{\alpha, \beta\} = \emptyset, \\ C_e \cup S_1, & \text{if } C'_e \cap \{\alpha, \beta\} = \{\alpha\}, \\ C_e \cup S_2, & \text{if } C'_e \cap \{\alpha, \beta\} = \{\beta\}, \\ C_e \cup S_1 \cup S_2, & \text{if } C'_e \cap \{\alpha, \beta\} = \{\alpha, \beta\}. \end{cases}$$

Lemma 3.4 *If $x_e = 2$, then $|X_i| = 2$, for some $i \in [3]$, and e labels an edge joining v_1 with v_2 in G.*

Proof: Assume this result fails. Hence $|X_i| = |X_j| = 1$, for a 2-subset $\{i, j\}$ of $[3]$. If $\{i, j\} = \{2, 3\}$, then, we may permute y_3 with z_3 in the graph to assume that e label the edge v_3v_4. That is, $C_e = \{e, y_2, y_3\}$. Hence $D_e - C = \{e\}$ and so $e \in \mathrm{cl}_M(C) - C$. We arrive at a contradiction to Lemma 3.1(iii) by taking $S = \{e\}$ because $D_e \cap S_3 = \{y_2\}$ and $D_e \cap S_4 = \{y_3\}$. Thus $1 \in \{i, j\}$. By symmetry, we may assume that $\{i, j\} = \{1, 2\}$, say $C_e = \{e, y_1, y_2\}$; that is, e labels the edge v_0v_3. Observe that $D_e - C = \{e, y_1\}$. Therefore e belongs to $E(H_1)$ because $A = \{y_1, z_1\} \subseteq E(H_1)$. As $C \cap [D_e - (\{a_2, c_2\} \cup S_1 \cup S_2)] = \{y_2\}$, it follows, by Lemma 3.1(i), that $\{e, y_1\}$ and $\{a_2, c_2\}$ do not cross with respect to C. Thus $D_e = C_e$ or $D_e = C_e \cup S_1 \cup S_2$. Observe that C_e cannot be a circuit of M, otherwise $C \triangle C_e = (C - y_2) \cup \{e, y_1\}$ is a 9-element circuit of M. Thus

$$\{e, y_1, y_2\} \cup S_1 \cup S_2 \text{ is a circuit of } M. \tag{3.7}$$

Observe that $C_e \triangle \{y_1, y_2, z_1, z_2\} = \{e, z_1, z_2\}$ is a circuit of M_1/T. Taking $\{e, z_1, z_2\}$ instead of C_e in the previous argument, (3.7) became

$$\{e, z_1, z_2\} \cup S_1 \cup S_2 \text{ is a circuit of } M. \tag{3.8}$$

By (3.7) and (3.8),

$$(\{e, y_1, y_2\} \cup S_1 \cup S_2) \triangle (\{e, z_1, z_2\} \cup S_1 \cup S_2) = \{y_1, y_2, z_1, z_2\} = A \cup S_3$$

is a cycle of M properly contained in $A \cup S_2 \cup S_3$; a contradiction. □

Lemma 3.5 *If $x_e = 3$, then $\{e, y_1, y_2, y_3\}$ or $\{e, z_1, y_2, y_3\}$ is a circuit of M_1/T. Moreover, M_1/T must have $\{e, y_1, y_2, y_3\}$ or $\{e, z_1, y_2, y_3\}$ as a circuit.*

Proof: By the choice of C_e and (3.6), we have that $|X_1| = |X_2| = |X_3| = 1$. The first part of the result follows provide we replace C_e by C'_e, where

$$C'_e = \begin{cases} C_e \triangle \{y_2, y_3, z_2, z_3\}, & \text{when } \{z_2, z_3\} \subseteq C_e, \\ C_e \triangle \{y_1, y_3, z_1, z_3\}, & \text{when } \{y_2, z_3\} \subseteq C_e, \\ C_e \triangle \{y_1, y_2, z_1, z_2\}, & \text{when } \{z_2, y_3\} \subseteq C_e. \end{cases}$$

Observe that $[M'/F]|(S_3 \cup S_4 \cup A) = M(G')$, where G' is the graph obtained from G by identifying v_1 with v_2. In particular S_3, S_4 and A are 2-circuits of M'/F. If $x_f \in \{1, 2\}$, for every $f \in E(M_1) - (S_3 \cup S_4 \cup A \cup T)$, then, by Lemma 3.4, each element of $E(M_1) - (T \cup F)$ is in parallel with some element of $\{y_1, y_2, y_3\}$ in M'/F. As $\{y_1, y_2, y_3\}$ is independent in M'/F, it follows that, for each $i \in [3]$, there is a rank-1 connected component N_i of M'/F such that $y_i \in E(N_i)$. Thus M'/F has only rank-1 connected components; a contradiction. Therefore there is $f \in E(M_1) - (S_3 \cup S_4 \cup A \cup T)$ such that $x_f = 3$ and the second part of this result follows from the first. □

Lemma 3.6 $x_e \neq 2$.

Proof: Assume that $x_e = 2$, for some $e \in E(M_1) - (S_3 \cup S_4 \cup A \cup T)$. By Lemma 3.4, we can take $C_e = \{e, y_i, z_i\}$, for some $i \in [3]$. If $j \neq i$ and $j \in [3]$, we can replace C_e by $C_e \triangle \{y_i, z_i, y_j, z_j\} = \{e, y_j, z_j\}$. Thus we may assume that $C_e = \{e, y_2, z_2\}$. By Lemma 3.5, there is $f \in E(M_1) - (S_3 \cup S_4 \cup A \cup T)$ such that $x_f = 3$. If possible, choose C_f such that $\{a_2, c_2\}$ and $D_f - C$ cross with respect to C. Assume that $C_f = \{f, y_1, y_2, y_3\}$. Observe that $f \in E(H_1)$ because $D_f - C = \{f, y_1\}$ and $y_1 \in E(H_1)$. Now, we prove that $\{a_2, c_2\}$ and $\{f, y_1\}$ does not cross with respect to C. If $\{a_2, c_2\}$ and $\{f, y_1\}$ cross with respect to C, then, by Lemma 3.1(i), $S_5 = \{y_2, y_3\} = D_f - (\{a_2, c_2\} \cup S_1 \cup S_2)$ and $S_6 = \{z_2, z_3\} = C - [D_f \cup (\{a_2, c_2\} \cup S_1 \cup S_2)]$ are series classes of $M|(C \cup \{f, y_1, a_2, c_2\})$. Note that D_e meets both S_5 and S_6 in just one element; a contradiction to Lemma 3.1(iii) because $e \in \mathrm{cl}_M(C) - C$. Thus $\{a_2, c_2\}$ and $\{f, y_1\}$ does not cross with respect to C. Now, $D_f \cap (S_1 \cup S_2) = \emptyset$ or $S_1 \cup S_2 \subseteq D_f$. Then $D_f = C_f$ or $D_f \triangle C = \{f, y_1, z_2, z_3\}$ is a circuit of M respectively, say C_f is a circuit of M. But $C_f \triangle (A \cup S_1 \cup S_4) = \{f, z_1\} \cup \{y_2, z_3\} \cup S_1$ is a circuit of M. Thus $\{a_2, c_2\}$ and $\{f, z_1\}$ cross with respect to C. Therefore $\{f, z_1\} \cup \{y_2, z_3\}$ is contrary to the choice of C_f. □

By Lemmas 3.5 and 3.6, the simplification K of M_1/T is isomorphic to F_7^* or $AG(3, 2)$. The following matrix gives the binary representation of K. The labels of the first 6 columns are respectively y_1, z_1, y_2, y_3, z_2 and z_3. At least one of the last two columns must exist in the representation of K.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Consequently M_1 satisfies (B) of Theorem 1.5(ii). Therefore the proof of Theorem 1.5 is concluded.

References

[1] C. Chun, J. Oxley and K. Wetzler, The binary matroids with no odd circuits of size exceeding five, *J. Combin. Theory Ser. B* 152 (2022), 80–120.

[2] R. Cordovil, B. Maia Jr. and M. Lemos, The 3-connected binary matroids with circumference 6 or 7, *European J. Combin.* 30 (2009), 1810–1824.

[3] R. Cordovil and M. Lemos, The 3-connected matroids with circumference 6, *Discrete Math.* 310 (2010), 1354–1365.

[4] M. Lemos, The 3-connected binary matroids with circumference 8, part II, in preparation.

[5] M. Lemos, The 3-connected binary matroids with circumference 8, part III, in preparation.

[6] M. Lemos and J. Oxley, On size, circumference and circuit removal in 3-connected matroids, *Discrete Math.* 220 (2000), 145–157.

[7] J. Oxley, *Matroid Theory, Second Edition*, Oxford University Press, New York, 2011.

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