

Existence of λ -fold non-zero sum Heffter arrays through local considerations

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Abstract

The paper [Costa et al., *Discrete Math.* 345 (2022), 112952] introduced, for cyclic groups, the class of partially filled arrays of the non-zero sum Heffter array that are, as the Heffter arrays, related to difference families, graph decompositions, and biembeddings. Here we generalize this definition to any finite group. Given a subgroup J of order t of a finite group $(G, +)$, a λ -fold non-zero sum Heffter array over G relative to J , ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$, is an $m \times n$ partially filled array with entries in G such that: each row contains h filled cells and each column contains k filled cells; for every $x \in G \setminus J$, the sum of the occurrences of x and $-x$ is λ ; the sum of the elements in every row and column is, following the natural orderings from left to right for the rows and from top to bottom for the columns, different from 0. The 2022 paper by Costa et al. also presented a complete, probabilistic solution for the existence problem in case $\lambda = 1$ and $G = \mathbb{Z}_v$ which is the starting point of this investigation. In this paper we will consider the existence problem for a generic value of λ and a generic finite group G , and we present an almost complete solution to this problem. In particular, we will prove, through local considerations (inspired by the Lovász Local Lemma), that there exists a λ -fold non-zero sum Heffter array over G relative to J whenever the trivial necessary conditions are satisfied and $|G| \geq 41$. This value can be reduced to 29 in case the array does not contain empty cells. Finally, we will show that these arrays give rise to biembeddings of multigraphs into orientable surfaces and we provide new infinite families of such embeddings.

1 Introduction

An $m \times n$ partially filled (p.f., for short) array on a set Ω is an $m \times n$ matrix whose elements belong to Ω and where some cells can be empty.

First, we need to introduce some notation. The rows and the columns of an $m \times n$ p.f. array A are denoted by R_1, \dots, R_m and by C_1, \dots, C_n , respectively. Here, we name by $\mathcal{E}(A)$, $\mathcal{E}(R_i)$, $\mathcal{E}(C_j)$ the list of the elements of the filled cells of A , of the i -th row and of the j -th column, respectively. Then we denote by $\text{skel}(A)$ the *skeleton* of A , that is the set of the filled positions of A . Also, when we define a multiset M we use the square brackets and we denote M^λ to be the multiset in which each element of M (counted with its multiplicity) is repeated λ times.

In 2015 Archdeacon [3] introduced a class of p.f. arrays which has been extensively studied: the *Heffter arrays*.

Definition 1.1 A *Heffter array* $H(m, n; h, k)$ is an $m \times n$ p.f. array with entries in the cyclic group $(\mathbb{Z}_{2nk+1}, +)$ such that:

- (a) each row contains h filled cells and each column contains k filled cells;
- (b) for every $x \in \mathbb{Z}_{2nk+1} \setminus \{0\}$, either x or $-x$ appears in the array;
- (c) the elements in every row and column sum to 0.

Trivial necessary conditions for the existence of an $H(m, n; h, k)$ are $mh = nk$, $3 \leq h \leq n$ and $3 \leq k \leq m$. If $m = n$ then $h = k$ and a square $H(n, n; k, k)$ will be simply denoted by $H(n; k)$. For those kinds of arrays, the existence problem has been completely solved. Indeed, in [4, 7, 17], the following result was proved.

Theorem 1.2 *An $H(n; k)$ exists for every $n \geq k \geq 3$.*

In [14] (see also [25]) this concept has been generalized as follows.

Definition 1.3 Let m, n, h, k be four positive integers. Let λ be a divisor of $2nk$ and t be a divisor of $\frac{2nk}{\lambda}$. Set $v = \frac{2nk}{\lambda} + t$ and let J be the subgroup of $(\mathbb{Z}_v, +)$ of order t . A λ -fold *Heffter array* A over \mathbb{Z}_v relative to J , denoted by ${}^\lambda H_{G,J}(m, n; h, k)$, is an $m \times n$ p.f. array with elements in \mathbb{Z}_v such that:

- (a₁) each row contains h filled cells and each column contains k filled cells;
- (b₁) the multiset $[\pm x \mid x \in \mathcal{E}(A)]$ contains each element of $\mathbb{Z}_v \setminus J$ exactly λ times;
- (c₁) the elements in every row and column sum to 0.

It is easy to see that if $\lambda = t = 1$ we again find the arrays of Definition 1.1.

Classical Heffter arrays and the above generalization have also been introduced because of their vast variety of applications and connections with other much-studied problems and concepts. In particular, in the paper of Archdeacon (and in other recent ones), Heffter arrays are investigated to obtain new face 2-colorable embeddings (briefly biembeddings). On the other hand, several authors focused their attention

on the existence problem of Heffter arrays and their variations. For a complete discussion on this topic, we refer the reader to the survey [27] and the references therein.

In [11], a new class of p.f. arrays was presented, which is related to that of Heffter arrays: the non-zero sum Heffter arrays. This definition was motivated by Alspach’s partial sums conjecture which deals with sets whose sums are different from 0; see [2, 5, 6, 20, 26]. Here we propose a generalization of this concept to a generic finite group (as done in [13] for the classical Heffter arrays).

Definition 1.4 Let $(G, +)$ be a finite group and let J be a subgroup of G . A λ -fold non-zero sum Heffter array A over G relative to J , denoted by ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$, is an $m \times n$ p.f. array with elements in G such that:

- (a₂) each row contains h filled cells and each column contains k filled cells;
- (b₂) the multiset $[\pm x \mid x \in \mathcal{E}(A)]$ contains λ times each element of $G \setminus J$;
- (c₂) the sum of the elements in every row, following the natural ordering from left to right, and column, following the natural ordering from top to bottom, is different from 0.

Here if $\lambda = 1$, $J = \{0\}$, $m = k$ or $m = n$, we will use, respectively, the notation $\text{NH}_{G,J}(m, n; h, k)$, ${}^\lambda\text{NH}_G(m, n; h, k)$, ${}^\lambda\text{NH}_{G,J}(m, n)$ and ${}^\lambda\text{NH}_{G,J}(n; k)$.

Example 1.5 Let G be the dihedral group of order 6. Its elements are, for definition, the pairs $(x, y) : x \in (\mathbb{Z}_3, +_3), y \in (\mathbb{Z}_2, +_2)$, and the group operation is defined as $(x_1, y_1) + (x_2, y_2) = (x_1 +_3 (\varphi_{y_1} x_2), y_1 +_2 y_2)$ where $\varphi_0 = 1$ and $\varphi_1 = -1$. Set $J = \mathbb{Z}_3 \times \{0\}$, we provide an example of a ${}^4\text{NH}_{G,J}(3; 2)$.

$$A := \begin{array}{|c|c|c|} \hline (1, 1) & (2, 1) & \\ \hline & (1, 1) & (0, 1) \\ \hline (0, 1) & & (2, 1) \\ \hline \end{array}$$

Indeed, note that $\mathcal{E}(A) = [(0, 1), (0, 1), (1, 1), (1, 1), (2, 1), (2, 1)]$ and hence

$$[\pm x \mid x \in \mathcal{E}(A)] = [(0, 1), (0, 1), (0, 1), (0, 1), (1, 1), (1, 1), (1, 1), (1, 1), (2, 1), (2, 1), (2, 1), (2, 1)]$$

covers $G \setminus J$ four times.

In [11], the authors focused on the case $\lambda = 1$, namely when the entries of the array are pairwise distinct and considered only cyclic groups (see also [9] and [21] for other results on the cyclic case). In particular, they proved the existence of an $\text{NH}_{\mathbb{Z}_v,J}(m, n; h, k)$, where J is the subgroup of order t of \mathbb{Z}_v , whenever the trivial necessary conditions $nk = mh$, $v = \frac{2nk}{\lambda} + t$ and $t \mid v$ are satisfied. We also recall the very recent paper [22] where the authors present complementary (and constructive) results on arrays with non-zero row and column sums.

In this paper, we will consider the existence problem for a generic value of λ and we present an almost complete solution to this problem. For this purpose, in Section 2, we will first determine more general necessary conditions for the existence problem from which a complete solution for $|G| = 2$ follows. Then, in Section 3, we will try to solve this problem with direct constructions. In particular, we will present a complete solution also when $|G| = 3$ and in the case of square arrays over abelian groups of odd order. In Section 4, we will revisit the construction of [11] proving that there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ whenever the necessary conditions are satisfied provided that h and k are big enough if λ is considered fixed. With this procedure, for any group G , we leave the problem open for an infinite number of arrays. This issue is due to the fact that, generating uniformly at random the matrix, the probability of the event that a given row (or a given column) sums to zero is not, in general, independent from all the other entries of the array. Then, inspired by the Lovász Local Lemma (see [18]), we impose a different distribution: we tessellate the array with suitable tiles and we choose the elements of a tile uniformly at random among a small prescribed set. Within these local considerations, we reduce the number of dependencies among the events mentioned above. It will follow that there exists a λ -fold non-zero sum Heffter array over G relative to J whenever the necessary conditions are satisfied and $|G| \geq 41$ leaving the problem open only for a finite set of groups. This value can be turned down to 29 in case the array does not contain empty cells. Finally, in the last section of this paper, we show that these arrays give rise to biembeddings of multigraphs into orientable surfaces and we provide infinite families of such embeddings.

2 Necessary conditions

In [11], the authors proved that an $\text{NH}_{\mathbb{Z}_v, J}(m, n; h, k)$, where J is the subgroup of order t of \mathbb{Z}_v , exists whenever the trivial necessary conditions $nk = mh$, $v = \frac{2nk}{\lambda} + t$ and $t \mid v$ are satisfied. Those conditions can be easily generalized to the following ones.

Remark 2.1 Given a subgroup J of a finite group G , a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ can exist only if the following necessary conditions are satisfied:

$$(*) \quad \begin{cases} nk = mh; \\ \lambda \mid 2nk; \\ |G| = \frac{2nk}{\lambda} + |J|; \\ \text{if } G \setminus J \text{ contains an involution, then } \lambda \text{ must be even.} \end{cases}$$

We note that the latter condition is automatically satisfied in the case of cyclic groups and we observe that $|J| \mid \frac{2nk}{\lambda}$ since by the Lagrange's theorem the order of J divides the order of G .

Now we will see that in the case of groups isomorphic to \mathbb{Z}_2^r those conditions are not sufficient. Indeed, for such groups, we can state more strict conditions. We begin by presenting the case $G = \mathbb{Z}_2$.

Theorem 2.2 For $m \geq k \geq 1$ and $n \geq h \geq 1$, there exists a ${}^\lambda\text{NH}_{\mathbb{Z}_2}(m, n; h, k)$ if and only if the necessary conditions (*) of Remark 2.1 hold and hk is odd.

Proof. Since we can fill the array only with the involution $1 \in \mathbb{Z}_2$, hk must be odd. On the other hand, it is easy to construct an $m \times n$ p.f. array that has exactly h filled cells in each row and k filled cells in each column. \square

Next, in the following theorem, we consider the case where G is isomorphic to \mathbb{Z}_2^r .

Theorem 2.3 Let $(G, +)$ be a finite group isomorphic to \mathbb{Z}_2^r and let J be a proper subgroup of G of order t that is isomorphic to $\mathbb{Z}_2^{r_1}$. Then there exists a ${}^\lambda\text{NH}_{G,J}(1, n)$ if and only if the necessary conditions (*) of Remark 2.1 are satisfied and one of the following conditions holds:

- 1) $t = 2$ (i.e. $r_1 = 1$) and n is an odd multiple of $|G| - 2$;
- 2) $|G| = 2$ (i.e. $r = 1$ and $r_1 = 0$) and n is odd.

Proof. It is easy to check (see Theorem 1.2 of [1] for a reference) that the sum of the elements of a group G which is isomorphic to \mathbb{Z}_2^r is non-zero only if $r = 1$. It follows that the sum of the elements of $G \setminus J$ is non-zero only when t or $|G|$ are equal to 2.

Let us assume that there exists ${}^\lambda\text{NH}_{G,J}(1, n)$ and let us denote it by A . Since $G \setminus J$ only contains involutions we have that:

$$\mathcal{E}(A) = \mathcal{E}(R_1) = (G \setminus J)^{\lambda/2}.$$

Clearly, A can exist only if the sum of the elements of $G \setminus J$ is non-zero. So we can assume that $t = 2$ or $|G| = 2$.

If $t = 2$, then the sum of the elements of $G \setminus J$ is non-zero but it is an involution of G ($G \setminus \{0\}$ only contains involution). It follows that

$$\sum_{a \in \mathcal{E}(R_1)} a = \lambda/2 \sum_{g \in G \setminus J} g \neq 0$$

only if $\lambda/2$ is odd, or equivalently, if n is an odd multiple of $|G| - t$.

The statement follows since the case $|G| = 2$ has already been considered in Theorem 2.2. \square

Example 2.4 Let $G = \mathbb{Z}_2^4$ and let J be the subgroup generated by $(0, 1, 1, 0)$, $(1, 0, 0, 0)$, $(1, 1, 1, 1)$. It is easy to see that J is isomorphic to \mathbb{Z}_2^3 and hence it has eight elements. We now wonder whether a ${}^2\text{NH}_{G,J}(1, 8)$ exists. Since the sum of the elements of both \mathbb{Z}_2^4 and J (which is isomorphic to \mathbb{Z}_2^3) is zero, we have

$$\sum_{g \in G \setminus J} g = \sum_{g \in G} g - \sum_{g \in J} g = 0.$$

It follows that there does not exist any ${}^2\text{NH}_{G,J}(1, 8)$.

Remark 2.5 It must be noted that, for $|G| = 2$ and for the arrays of just one row/column, the necessary conditions $(*)$ of Remark 2.1 are not sufficient. Theorems 2.2 and 2.3 are the only theorems presented in this paper where we need more restrictive conditions. Hence we believe that the smallest groups G (and the arrays with just one row/column) are, somehow, intrinsically different from the general case.

3 Some direct constructions

The goal of this section would have been to solve this problem with direct constructions. Even though we fail to achieve this goal we are able to present a complete direct solution when $|G| = 3$ and in the case of square arrays over abelian groups of odd order.

3.1 A complete solution for $G = \mathbb{Z}_3$

Theorem 3.1 *There exists a ${}^\lambda\text{NH}_{\mathbb{Z}_3}(m, n; h, k)$ whenever the necessary conditions $(*)$ are satisfied.*

Proof. Since $nk = mh$ we have that $r := \frac{nk}{\text{lcm}(m,n)}$ is an integer. Then we have that also the following quantities are integers

$$\frac{k}{r} = \frac{\text{lcm}(m, n)}{n} \quad \text{and} \quad \frac{h}{r} = \frac{\text{lcm}(m, n)}{m}.$$

Hence we can define the following set of cells

$$Q := \{(i, j) : 1 \leq i \leq k/r; 1 \leq j \leq h/r\}.$$

Then, since

$$\frac{r \cdot n}{h} = \frac{r \cdot m}{k} = \frac{mn}{\text{lcm}(m, n)} = \text{gcd}(m, n),$$

we can consider the subset of the $m \times n$ array defined by:

$$B := \bigcup_{j=0}^{r-1} \left(\bigcup_{i=0}^{rn/h-1} Q + j(0, h/r) + i(k/r, h/r) \right)$$

where $Q + (x, y)$ is defined as $\{(i + x, j + y) \mid (i, j) \in Q\}$ and all the arithmetic on the row and column indices is performed modulo m and n respectively and the sets of reduced residues are $\{1, 2, \dots, m\}$ and $\{1, 2, \dots, n\}$.

Here, we fill all the cells of B with 1's obtaining the array A_1 . Now, if $3 \mid k$ then we replace all the elements in the set of cells H_1 by -1 's where the indices are considered modulo m and where

$$H_1 := \bigcup_{i=1}^{\lceil n/h \rceil} \text{skel}(R_{ik}),$$

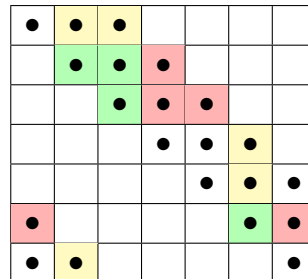
where R_{ik} is the (ik) -th row of A_1 . We denote by A_2 the array so defined. Here, if 3 does not divide k we set $A_2 = A_1$. Since every column intersects the rows of H_1 in either 1 or 2 cells, the sum of the elements in any given column of A_2 is non-zero.

Similarly, if $3 \mid h$ then we change the sign to all elements of A_2 in the set of cells H_2 , i.e. the 1's are replaced by -1 's and vice versa, where

$$H_2 := \bigcup_{j=1}^{\lceil m/k \rceil} \text{skel}(C_{jh}),$$

where C_{jh} is the (jh) -th column of A_2 . We denote by A the array so defined. Here, if 3 does not divide h we set $A = A_2$. Since every row intersects the columns of H_2 in either one or two cells, the sum of the elements in any given row of A is non-zero. Moreover, since every column of A intersects H_2 in either 0 or k cells and the sum of the elements in any given column of A_2 is non-zero, also the sum of all the elements in any column of A is non-zero. Hence, we obtain a ${}^\lambda\text{NH}_{\mathbb{Z}_3}(m, n; h, k)$ such that $\text{skel}(A) = B$. \square

Example 3.2 In the following, we provide an example of the sets B , H_1 , and H_2 for $m = n = 7$ and $k = h = 3$. The black dots represent the cells contained in the set B , the green cells are the ones contained in $H_1 \cap H_2$, the red cells are the ones contained in $H_1 \setminus H_2$ and finally, the yellow cells are the ones in $H_2 \setminus H_1$.



Then, applying the proof of Theorem 3.1 we get the following ${}^{21}\text{NH}_{\mathbb{Z}_3}(7; 3)$.

1	-1	-1				
	1	1	-1			
		1	-1	-1		
			1	1	-1	
				1	-1	1
-1					1	-1
1	-1					1

Example 3.3 Applying the proof of Theorem 3.1 we obtain the following ${}^{12}\text{NH}_{\mathbb{Z}_3}(4, 6; 3, 2)$.

1	1	-1			
1	1	-1			
			1	1	-1
			1	1	-1

3.2 On abelian groups G of odd order

Theorem 3.4 *Let $(G, +)$ be a finite abelian group of odd order and let J be a subgroup of G . For $n \geq k \geq 1$, there exists a ${}^\lambda\text{NH}_{G,J}(n; k)$ whenever the necessary conditions $(*)$ are satisfied.*

Proof. Since $|G|$ is odd, also $|J|$ must be odd: so, we can write $|G \setminus J| = 2w$. First, we consider the case $w \geq 2$.

Let Q be the set of cells given by $Q := \{(i, 1) : 1 \leq i \leq k\}$. We consider the subset of the $n \times n$ array defined by:

$$B := \bigcup_{i=0}^{n-1} (Q + i(1, 1))$$

where the sum is considered modulo n and the set of reduced residues is $\{1, 2, \dots, n\}$.

Let us order the elements of $G \setminus J$ as g_1, \dots, g_{2w} with the additional property that if $x \in \{g_1, \dots, g_w\}$ then $-x \in \{g_{w+1}, \dots, g_{2w}\}$. We consider the multiset $\Omega = \{g_1, \dots, g_w\}^\lambda$ and an array A_1 such that $\text{skel}(A_1) = B$ and $\mathcal{E}(A_1) = \Omega$.

Clearly, if $k = 1$, however we fill the cells of B in the array with the elements of Ω , we obtain a ${}^\lambda\text{NH}_{G,J}(n; k)$. So we may assume that $k \geq 2$. In this case, for each $i \in [1, n]$, we denote by y_i and x_i the elements of the i -th column C_i that are in positions (i, i) , $(i + 1, i)$ respectively (where the sum is considered modulo n). Since $w \geq 2$, we can choose y_i in such a way that $y_i \pm x_i \neq 0$. It suffices to set $x_i = g_{2i+1}$ and $y_i = g_{2i}$ where the indices are considered modulo w and then to choose, arbitrarily among the remaining elements of Ω , the rest of the array.

We claim that there exists a choice between x_i and $-x_i$ such that the sum of the elements in the i -th column is different from zero either using y_i or $-y_i$.

Indeed, if x_i does not have this property, we have that either

$$\left(\sum_{a \in \mathcal{E}(C_i) \setminus \{x_i, y_i\}} a \right) + x_i + y_i = 0 \tag{3.1}$$

or

$$\left(\sum_{a \in \mathcal{E}(C_i) \setminus \{x_i, y_i\}} a \right) + x_i - y_i = 0. \tag{3.2}$$

If equation (3.1) holds then we have that

$$\left(\sum_{a \in \mathcal{E}(C_i) \setminus \{x_i, y_i\}} a \right) - x_i + y_i \neq 0.$$

Moreover, since the group G is an abelian group of odd order and since $y_i \neq -x_i$ by construction, we have that

$$x_i + y_i \neq -(x_i + y_i) = -x_i - y_i,$$

where in the last equality we used the fact that G is abelian. Hence we also have that

$$\left(\sum_{a \in \mathcal{E}(C_i) \setminus \{x_i, y_i\}} a \right) - x_i - y_i \neq 0.$$

If equation (3.2) holds, instead, we have that

$$\left(\sum_{a \in \mathcal{E}(C_i) \setminus \{x_i, y_i\}} a \right) - x_i - y_i \neq 0.$$

Moreover, as done before, since the group G is abelian of odd order and since $y_i \neq x_i$ by construction, we also have that

$$\left(\sum_{a \in \mathcal{E}(C_i) \setminus \{x_i, y_i\}} a \right) - x_i + y_i \neq 0.$$

Based on this simple argument, we can assume that the element x_i in position $(i + 1, i)$ is such that the sum of the elements in the i -th column C_i is different from zero either using y_i or $-y_i$. We denote by A_2 the array so defined. Then, we replace the element y_i in position (i, i) of A_2 by $-y_i$ for each row $R_i \in [1, n]$ that sums to zero. It follows that we obtain a ${}^\lambda\text{NH}_{G,J}(n; k)$ whenever $w \geq 2$.

We now observe that, if $|G| \geq 5$ is odd, then $|G \setminus J| \geq 4$ for any proper subgroup J of G . The statement follows since the case $G = \mathbb{Z}_3$ has already been considered in Theorem 3.1. □

Example 3.5 Let $G = \mathbb{Z}_{13}$ and let $J = \{0\}$ (i.e. $t = 1$). We are wondering whether there exists a ${}^2\text{NH}_{G,J}(4; 3)$. Following the proof of Theorem 3.4, we define

$$\Omega = [1, 2, 3, 4, 5, 6]^2$$

and

$$A_2 := \begin{array}{|c|c|c|c|} \hline 1 & & 3 & 5 \\ \hline 2 & 4 & & -6 \\ \hline 3 & 5 & 1 & \\ \hline & -6 & 2 & 4 \\ \hline \end{array}$$

Here we have that $x_1 = 2, x_2 = 5, x_3 = 2, x_4 = 5$ and $y_1 = 1, y_2 = 4, y_3 = 1, y_4 = 4$ and the x_i 's are such that the sum of the elements in the i -th column C_i is non-zero either using y_i or $-y_i$.

Then, since the sum of the elements of R_2 and R_4 is zero, we replace the elements y_2 and y_4 with $-y_2$ and $-y_4$. Finally, we obtain the following ${}^2\text{NH}_{G,J}(4; 3)$:

$$A := \begin{array}{|c|c|c|c|} \hline 1 & & 3 & 5 \\ \hline 2 & -4 & & -6 \\ \hline 3 & 5 & 1 & \\ \hline & -6 & 2 & -4 \\ \hline \end{array}$$

4 Our Method

The direct constructions presented in the previous section solve only some particular instances of the existence problem. On the other hand, we do not believe it would be possible to adapt these ideas to the general case since in both Theorem 3.1 and Theorem 3.4 we use the fact that the group G is involution free. Therefore, in order to attack this existence problem, we now develop a local variant (inspired by the Lovász Local Lemma) of the probabilistic method used in [11].

4.1 Bound with Expected values

First, we revisit the method of [11] in order to obtain some sufficient conditions on the existence of a λ -fold non-zero sum Heffter array.

Theorem 4.1 *Let $(G, +)$ be a group and let J be a subgroup of G . Then there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ if the necessary conditions $(*)$ are satisfied and the following inequality holds*

$$\lambda \left(\frac{m}{mh - h + 1} + \frac{n}{nk - k + 1} \right) < 1.$$

Proof. Let Ω be a multiset of elements of G whose size is nk and such that $\pm\Omega = (G \setminus J)^\lambda$. We denote by B the set of cells of an $m \times n$ partially filled array A having exactly h filled cells in each row and exactly k filled cells in each column that exists due to Lemma 4.1 of [11]. Note that here we have $\text{skel}(A) = B$.

We prove that we can fill the cells of B in the array A with the elements of Ω in such a way that the sum over each row and each column is non-zero. Let us pick uniformly at random the elements of A among the multiset Ω . In this way, we will ensure that $\mathcal{E}(A) = \Omega$. We denote by X_i the event that the i -th row sums to zero and by $\mathbb{P}(X_i)$ its probability. Also, we name by X the random variable given by the number of rows that sum to zero and by $\mathbb{E}(X)$ its expected value.

Due to the linearity of the expected value, we have that:

$$\mathbb{E}(X) = \sum_{i=1}^m \mathbb{P}(X_i).$$

By symmetry, $\mathbb{P}(X_i) = \mathbb{P}(X_1)$ for every $i = 1, \dots, m$, and hence

$$\mathbb{E}(X) = m \cdot \mathbb{P}(X_1).$$

If we have already chosen, following the natural ordering from left to right, the first $h - 1$ elements, x_1, \dots, x_{h-1} of the first row then there exist at most λ elements $\bar{x} \in \Omega \setminus \{x_1, \dots, x_{h-1}\}$ that make the sum zero. This means that

$$\mathbb{P}(X_1) \leq \frac{\lambda}{mh - (h - 1)}$$

and thus

$$\mathbb{E}(X) \leq \frac{\lambda m}{mh - (h - 1)}.$$

Analogously, if we denote by $\mathbb{E}(Y)$ the expected value of the random variable Y given by the number of columns that sum to zero then we obtain

$$\mathbb{E}(Y) \leq \frac{\lambda n}{nk - (k - 1)}.$$

Therefore we can fill the cells of B in the array A with the elements of Ω such that the sum over each row and column is non-zero whenever $\mathbb{E}(X) + \mathbb{E}(Y) < 1$. \square

Remark 4.2 Note that the inequality in Theorem 4.1 is satisfied when the parameters h and k are sufficiently large if λ is considered fixed.

In the next sections, we improve the result of Theorem 4.1 by proving that there exists a λ -fold non-zero sum Heffter array over G relative to J for $|G|$ larger than a small constant that does not depend on λ .

4.2 Tiling in LLL style

The procedure of the previous paragraph, for any group G , leave the problem open for an infinite number of arrays. This problem is caused by the fact that, generating the matrix uniformly at random, the probability of the event X_i (or Y_i) that a given row (or a given column) sums to zero depends on all the entries of the array. Here, inspired by the Lovász Local Lemma (see [18]), we impose a different distribution: we tessellate the array with suitable tiles and we choose the elements of a tile uniformly at random among a small prescribed set. With these local considerations, we reduce the number of dependencies among the events.

First, we define the set of admissible tiles we will use as bricks in our construction.

Definition 4.3 A set T of cells in a $m \times n$ array is said to be a *nice tile* if, for any group G , for any subset S of G of size $|T|$, and for any vectors (r_1, \dots, r_m) and (c_1, \dots, c_n) of, respectively, G^m and G^n , there exists an $m \times n$ partially filled array A such that:

- (a) $\text{skel}(A) = T$;
- (b) $\mathcal{E}(A) = S$;
- (c₁) if the i -th row of A is non-empty, there exist $\beta_i \in [1, n]$ such that

$$\text{skel}(R_i) = \{(i, \beta_i), (i, \beta_i + 1), \dots, (i, \beta_i + h_i - 1)\}$$

where $h_i = |T \cap \text{skel}(R_i)|$ and the indices are taken modulo n ;

(c_2) if the j -th column of A is non-empty, there exist $\gamma_j \in [1, m]$ such that

$$\text{skel}(C_j) = \{(\gamma_j, j), (\gamma_j + 1, j), \dots, (\gamma_j + k_j - 1, j)\}$$

where $k_j = |T \cap \text{skel}(C_j)|$ and the indices are taken modulo m ;

(d_1) if the i -th row of A is non-empty, the sum of its elements, following the natural ordering from left to right and starting from the cell (i, β_i) , is different from r_i ;

(d_2) if the j -th column of A is non-empty, the sum of its elements, following the natural ordering from top to bottom and starting from the cell (γ_j, j) , is different from c_j .

Theorem 4.4 *Let B be the skeleton of a $m \times n$ array A that has exactly h filled cells in each row and k in each column. Then, given a finite group $(G, +)$ and a subgroup J of G , there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ whenever the necessary conditions $(*)$ are satisfied and:*

- 1) *it is possible to partition B into nice tiles T_1, \dots, T_ℓ ;*
- 2) $|G \setminus J| \geq \max_i(|T_i|)$.

Proof. We first assume that λ is even. In this case we note that $\Omega = (G \setminus J)^{\lambda/2}$ is such that $\pm\Omega = (G \setminus J)^\lambda$. Moreover, since the necessary conditions are satisfied, and since the tiles partition B , we have that

$$\frac{\lambda}{2}|G \setminus J| = nk = \sum_{i=1}^{\ell} |T_i|.$$

Now we enumerate the u elements of $G \setminus J$ as g_1, \dots, g_u . Here we want to define, recursively, the sets S_1, \dots, S_ℓ that we will use to fill the cells of the tiles T_1, \dots, T_ℓ . With abuse of notation, we will first define S_1, \dots, S_ℓ as ordered lists of non-repeated elements, and then we will consider the associated sets.

We set the list S_1 to be $(g_1, \dots, g_{|T_1|})$.

Given S_1, \dots, S_i , if $i < \ell$, we define S_{i+1} as follows. Naming g_j as the last element of the list S_i , we set

$$S_{i+1} := (g_{j+1}, \dots, g_{j+|T_{i+1}|})$$

where the indices are considered modulo u .

Here we note that, since $u \geq \max_i(|T_i|)$, the lists S_1, \dots, S_ℓ do not have repeated elements and hence we can consider them to be sets.

Now we fill, recursively, the tiles T_1, \dots, T_ℓ with the elements of S_1, \dots, S_ℓ .

For the tile T_1 we denote by \mathcal{R}_1 the set of the rows of A such that $R_i \in \mathcal{R}_1$ whenever $(B \setminus T_1) \cap \text{skel}(R_i)$ is empty. Set $I_1 = \{i \in [1, m] : R_i \in \mathcal{R}_1\}$, we consider a vector $(r_1, \dots, r_m) \in G^m$ such that $r_i = 0$ for any $i \in I_1$. Similarly, we consider the set \mathcal{C}_1 of the columns of A such that $C_j \in \mathcal{C}_1$ whenever $(B \setminus T_1) \cap \text{skel}(C_j)$ is empty. Set $J_1 = \{j \in [1, n] : C_j \in \mathcal{C}_1\}$. We consider a vector $(c_1, \dots, c_n) \in G^n$ such that $c_j = 0$ for any $j \in J_1$. Since T_1 is a nice tile, we can fill it with the elements of S_1 , obtaining an array A_1 such that

- (d₁) if the i -th row of A_1 is non-empty, the sum of its elements, following the natural ordering from left to right, is different from r_i ;
- (d₂) if the j -th column of A_1 is non-empty, the sum of its elements, following the natural ordering from top to bottom, is different from c_j .

Now we assume we have defined the arrays A_1, \dots, A_b . Then, if $b < \ell$, we define the array A_{b+1} as follows. First, we denote by \bar{A}_b the union $\bigcup_{i=1}^b A_i$ that is the $m \times n$ array in which we have joined together all the non-empty elements of A_1, \dots, A_b . Then by $\bar{R}_1, \dots, \bar{R}_m$ we denote the rows of \bar{A}_b and by $\bar{C}_1, \dots, \bar{C}_n$ its columns. Then we consider the set \mathcal{R}_{b+1} of the rows of A such that $(B \setminus \text{skel}(\bar{A}_b)) \cap \text{skel}(R_i)$ is non empty but $(B \setminus (\text{skel}(\bar{A}_b) \cup T_{b+1})) \cap \text{skel}(R_i)$ is empty. Set $I_{b+1} = \{i \in [1, m] : R_i \in \mathcal{R}_{b+1}\}$, and given $i \in I_{b+1}$, we have that, because of property (c₁) of Definition 4.3, the cells of $B \cap \text{skel}(R_i)$ are either of the form

$$\{(i, \alpha_{h'_{i+1}}), \dots, (i, \alpha_{h'_i}), (i, \beta_i), (i, \beta_i + 1), \dots, (i, \beta_i + (h_i - 1)), (i, \alpha_1), \dots, (i, \alpha_{h'_{i+1}})\}$$

or

$$\{(i, \beta_i + h_{i+1}), \dots, (i, \beta_i + (h_i - 1)), (i, \alpha_1), (i, \alpha_2), \dots, (i, \alpha_{h'_i}), (i, \beta_i), \dots, (i, \beta_i + (h_{i+1} - 1))\}$$

where the α 's appears in the cells of $\text{skel}(\bar{A}_b)$ and the β 's in those of T_{b+1} . In the following, we will denote by $\bar{a}_{i,j}$ the element of \bar{A}_b in position (i, j) . Here, given $i \in I_{b+1}$, we consider a vector $(r_1, \dots, r_m) \in G^m$ such that $r_i = -\sum_{j=1}^{h'_i} \bar{a}_{(i,\alpha_j)}$ where the sum is taken, starting from the cell (i, α_1) , following the natural ordering from left to right.

Similarly, we consider the set \mathcal{C}_{b+1} of the columns of A such that $(B \setminus \text{skel}(\bar{A}_b)) \cap \text{skel}(C_j)$ is non empty but $(B \setminus (\text{skel}(\bar{A}_b) \cup T_{b+1})) \cap \text{skel}(C_j)$ is empty. Set $J_{b+1} = \{j \in [1, n] : C_j \in \mathcal{C}_{b+1}\}$, and given $j \in J_{b+1}$, we have that, because of property (c₂) of Definition 4.3, the filled cells of $\text{skel}(C_j)$ are either of the form

$$\{(\alpha_{k'_{j+1}}, j), \dots, (\alpha_{k'_j}, j), (\gamma_j, j), (\gamma_j + 1, j), \dots, (\gamma_j + (k_j - 1), j), (\alpha_1, j), \dots, (\alpha_{k'_{j+1}}, j)\}$$

or

$$\{(\gamma_j + k_{j+1}, j), \dots, (\gamma_j + (k_j - 1), j), (\alpha_1, j), (\alpha_2, j), \dots, (\alpha_{k'_j}, j), (\gamma_j, j), \dots, (\gamma_j + (k_{j+1} - 1), j)\}$$

where the α 's appears in the cells of $\text{skel}(\bar{A}_b)$ and the γ 's in those of T_{b+1} . Here, given $j \in J_{b+1}$, we consider a vector $(c_1, \dots, c_n) \in G^n$ such that $c_j = -\sum_{i=1}^{k'_j} \bar{a}_{(\alpha_i, j)}$ where, starting from the cell (α_1, j) , the sum is taken following the natural ordering from top to bottom. Since T_{b+1} is a nice tile, we can fill it with the elements of S_{b+1} , obtaining an array A_{b+1} such that

- (d₁) if the i -th row of A_{b+1} is non-empty, the sum of its elements, following the natural ordering from left to right, is different from r_i ;
- (d₂) if the j -th column of A_{b+1} is non-empty, the sum of its elements, following the natural ordering from top to bottom, is different from c_j .

Here we note that, given a row $R_i \in \mathcal{R}_{b+1}$ with $i \in I_{b+1}$, because of property (c_1) of Definition 4.3, $\mathcal{E}(R_i \cap \bar{A}_{b+1})$ is, following the natural ordering, either of the form

$$(\bar{a}_{h'_{i_1}+1}, \dots, \bar{a}_{h'_i}, a_1, a_2, \dots, a_{h_i}, \bar{a}_1, \dots, \bar{a}_{h'_{i_1}})$$

or

$$(a_{h_{i_1}+1}, \dots, a_{h_i}, \bar{a}_1, \bar{a}_2, \dots, \bar{a}_{h'_i}, a_1, \dots, a_{h_{i_1}})$$

where the \bar{a} 's belong to \bar{A}_b and the a 's are elements of A_{b+1} .

In both cases, we have that

$$a_1 + \dots + a_{h_i} \neq -(\bar{a}_1 + \dots + \bar{a}_{h'_i}) = -\bar{a}_{h'_i} - \dots - \bar{a}_1.$$

This implies, in the first case that

$$\bar{a}_{h'_{i_1}+1} + \dots + \bar{a}_{h'_i} + a_1 + a_2 + \dots + a_{h_i} + \bar{a}_1 + \dots + \bar{a}_{h'_{i_1}} \neq 0$$

and, in the latter, that

$$a_{h_{i_1}+1} + \dots + a_{h_i} + \bar{a}_1 + \bar{a}_2 + \dots + \bar{a}_{h'_i} + a_1 + \dots + a_{h_{i_1}} \neq 0.$$

This means that the sum of the elements of R_i is, in both cases, non-zero for any $i \in I_{b+1}$.

Reasoning in the same way also for the columns, we obtain that the sum of the elements of C_j is non-zero whenever $j \in J_{b+1}$.

Therefore, we have that $\bigcup_{i=1}^{\ell} A_i$ defines a ${}^{\lambda}\text{NH}_{G,J}(m, n; h, k)$. Finally, we assume that λ is odd. Here, since the necessary conditions are satisfied, we do not have involutions in $G \setminus J$ and we note that if $x \in G \setminus J$ then also $-x$ is in $G \setminus J$. It follows that $|G \setminus J|$ is even and we can enumerate the elements of $G \setminus J$ as g_1, \dots, g_{2w} with the additional property that if $x \in \{g_1, \dots, g_w\}$ then $-x \in \{g_{w+1}, \dots, g_{2w}\}$.

Here we note that

$$\Omega = (G \setminus J)^{(\lambda-1)/2} \cup \{g_1, \dots, g_w\}$$

is such that $\pm\Omega = (G \setminus J)^{\lambda}$. Moreover, proceeding as in the case λ even, we have that

$$\frac{\lambda-1}{2}|G \setminus J| + \frac{1}{2}|G \setminus J| = \frac{\lambda}{2}|G \setminus J| = nk = \sum_{i=1}^{\ell} |T_i|.$$

It follows that we can partition Ω with lists S_1, \dots, S_{ℓ} that do not have repeated elements and such that $|S_i| = |T_i|$. Then, proceeding as in the case λ even, we obtain the statement also in the odd case. □

Remark 4.5 The procedure of the previous theorem reduces the dependencies among the events X_i 's (respectively Y_i 's) and hence the edges in the dependencies graph (see [18]). Indeed the event X_i , that the i -th row sums to zero, only depends on the tiles that intersect R_i .

Here our first aim was to apply the Lovász Local Lemma (see [18]) to solve some instances of the existence problem. However, since we only use nice tiles, even these dependencies do not affect the procedure of Theorem 4.4, so we can fill the array in a very general context, by simply acting locally, tile by tile.

5 Totally Filled Arrays

The goal of this section is to prove that, if we consider a totally filled $m \times n$ rectangular array B , we can define a ${}^\lambda\text{NH}_{G,J}(m, n)$ whose cells are those of B whenever the necessary conditions $(*)$ are satisfied and $|G| \geq 30$. First, we consider the case $m = 1$.

Proposition 5.1 *Let $(G, +)$ be a finite group and let J be a subgroup of G . Then there exists a ${}^\lambda\text{NH}_{G,J}(1, n)$ whenever the necessary conditions $(*)$ (and those of Theorem 2.3 in case G is isomorphic to \mathbb{Z}_2^r) are satisfied.*

Proof. We may assume G is not isomorphic to \mathbb{Z}_2^r since this case has been already considered in Theorem 2.3. This means that G is either non-abelian or isomorphic to $H \oplus \mathbb{Z}_2^{r_1}$ where H is a non-trivial abelian group of odd order and $r_1 \geq 0$.

CASE 1: G is non-abelian. We claim that, in this case, we have two elements x, y in $G \setminus J$ such that $x + y \neq y + x$. It is a well-known fact that a finite group cannot be written as the union of two proper subgroups. It follows that there is $y \in G \setminus J$ that is not in the center of G . Here we have that the centralizer $C_G(y)$ of y is a proper subgroup of J and hence, reasoning as before, there exists $x \in G \setminus J$ that is not in the centralizer of y and, in particular $x + y \neq y + x$ and $x \neq \pm y$.

Now we order the u elements of $G \setminus J$ as g_1, \dots, g_u so that $g_1 = x$ and $g_2 = y$. Moreover, if λ is odd, we have that $G \setminus J$ does not contain any involution and thus we can assume that $\pm\{g_1, \dots, g_{\frac{u}{2}}\} = G \setminus J$ as done in the proof of Theorem 4.4. In the following, we denote by B the set of cells of a $1 \times n$ totally filled array. We fill the elements of B so that, in position $(1, i)$ we put the element g_i where the index i is considered modulo u . Denoting by A_1 the array so defined, we have that its element in position $(1, i)$ is $a_{1,i} = g_{i \pmod u}$.

Since the necessary conditions are satisfied, we have filled B with the elements of:

$$\Omega := \begin{cases} (G \setminus J)^{\lambda/2} & \text{if } \lambda \equiv 0 \pmod{2}; \\ (G \setminus J)^{(\lambda-1)/2} \cup \{g_1, \dots, g_{\frac{u}{2}}\} & \text{otherwise.} \end{cases}$$

In both cases, we have that the elements of $\pm\mathcal{E}(A_1) = \pm\Omega$ cover $G \setminus J$ exactly λ times.

Clearly, the sum over every column of A_1 is non-zero so it is left to prove that we can impose the sum over the first row to be non-zero. Let us assume that

$$\sum_{a \in \mathcal{E}(R_1)} a = x + y + \sum_{i=3}^n a_{1,i} = 0. \tag{5.1}$$

Indeed, if this equation does not hold, A_1 is already a ${}^\lambda\text{NH}_{G,J}(1, n)$. If equation (5.1) holds, since $x + y \neq y + x$ we must have that

$$y + x + \sum_{i=3}^n a_{1,i} \neq 0.$$

Therefore, if necessary, it suffices to switch x and y to obtain an array A that is a ${}^\lambda\text{NH}_{G,J}(1, n)$.

CASE 2: Let G be isomorphic to $H \oplus \mathbb{Z}_2^{r_1}$ where H is a non-trivial abelian group of odd order. In this case we have that $G \setminus (\{0\} \oplus \mathbb{Z}_2^{r_1})$ does not contain involutions and has size at least $\frac{2|G|}{3} > |G|/2$. Since $G \setminus J$ has size at least $|G|/2$, there exists a non involution element x in $G \setminus J$.

Now we order the elements of $G \setminus J$ as we did in CASE 1 and we assume that $g_1 = x$. Also here we denote by B the set of cells of a $1 \times n$ totally filled array and we fill the elements of B so that, in position $(1, i)$ we put the element g_i where the index i is considered modulo u . Denoting by A_1 the array so defined, we have that its element in position $(1, i)$ is $a_{1,i} = g_{i \pmod{u}}$. Also in this case, we have that $\pm\mathcal{E}(A_1)$ covers $G \setminus J$ exactly λ times.

Clearly, the sum over every column of A_1 is non-zero so it is left to prove that we can impose the sum over the first row to be non-zero. Let us assume that

$$\sum_{a \in \mathcal{E}(R_1)} a = x + \sum_{i=2}^n a_{1,i} = 0. \tag{5.2}$$

Indeed, if this equation does not hold, A_1 is already a ${}^\lambda\text{NH}_{G,J}(1, n)$. If equation (5.2) holds, since $-x \neq x$ we must have that

$$-x + \sum_{i=2}^n a_{1,i} \neq 0.$$

Therefore, in this case, it suffices to change x with $-x$ to obtain an array A that is a ${}^\lambda\text{NH}_{G,J}(1, n)$. □

Example 5.2 Let $G = \mathbb{Z}_3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ and let $J = \{0\} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. We are wondering if there exists a ${}^2\text{NH}_{G,J}(1, 8)$. Following the proof of Proposition 5.1, we set $x = (1, 0, 0)$ and we define

$$\Omega = [(1, 0, 0), (1, 1, 0), (1, 0, 1), (1, 1, 1), (2, 0, 0), (2, 1, 0), (2, 0, 1), (2, 1, 1)].$$

Here we note that $\sum_{g \in \Omega} g = 0$. Therefore, if we change the sign of x , we obtain the following ${}^2\text{NH}_{G,J}(1, 8)$:

(2, 0, 0)	(1, 1, 0)	(1, 0, 1)	(1, 1, 1)	(2, 0, 0)	(2, 1, 0)	(2, 0, 1)	(2, 1, 1)
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Now we list two families of nice tiles that we will use as bricks to fill the rectangular arrays.

Proposition 5.3 *Let T be a set of cells of an $m \times n$ array that satisfies the following conditions:*

- 1) T has exactly 3 non-empty rows $R_\alpha, R_{\alpha+1}, R_{\alpha+2}$ where the indices are considered modulo m ;

2) Each row $R_\alpha, R_{\alpha+1}, R_{\alpha+2}$ is non-empty exactly in the cells of the $b \geq 3$ columns $C_\beta, C_{\beta+1}, \dots, C_{\beta+(b-1)}$ where the indices are considered modulo n .

Then T is nice.

Proof. Let us consider a set S of $|T| = 3b$ elements of a finite group G and vectors $(r_1, \dots, r_m) \in G^m, (c_1, \dots, c_n) \in G^n$. Now we choose, uniformly at random, an array A such that $\text{skel}(A) = T$ and $\mathcal{E}(A) = S$. Following the notation of Theorem 4.1, we denote by $\mathbb{P}(X_i)$ the probability of the event X_i that the i -th row sums to r_i . Here, if $i \in \{\alpha, \alpha + 1, \alpha + 2\}$, we have that, chosen the first $b - 1$ elements (following the natural ordering) $a_{i,\beta}, \dots, a_{i,\beta+(b-2)}$ of the i -th row, there is at most one element $x \in S$ such that

$$a_{i,\beta} + \dots + a_{i,\beta+(b-2)} + x = r_i.$$

It follows that, if $i \in \{\alpha, \alpha + 1, \alpha + 2\}$,

$$\mathbb{P}(X_i) \leq \frac{1}{|S| - (b - 1)} = \frac{1}{2b + 1}.$$

Now we denote by $\mathbb{E}(X)$ the expected value of the random variable X given by the number of rows R_i , with $i \in \{\alpha, \alpha + 1, \alpha + 2\}$, that sums to r_i . Due to the linearity of the expected value, we have that:

$$\mathbb{E}(X) = \mathbb{P}(X_\alpha) + \mathbb{P}(X_{\alpha+1}) + \mathbb{P}(X_{\alpha+2}) \leq \frac{3}{2b + 1} < \frac{1}{2}.$$

Similarly, if we denote by $\mathbb{E}(Y)$ the expected value of the random variable Y given by the number of columns C_j , with $j \in \{\beta, \beta + 1, \dots, \beta + (b - 1)\}$, that sums to c_j , we have that:

$$\mathbb{E}(Y) \leq \frac{b}{3b - 2} < \frac{1}{2}.$$

Since

$$\mathbb{E}(X) + \mathbb{E}(Y) < 1$$

there exists an array A whose skeleton is T and such that $\mathcal{E}(A) = S$ and:

- (d_1) if the i -th row of A is non-empty, the sum of its elements is different from r_i ;
- (d_2) if the j -th column of A is non-empty, the sum of its elements is different from c_j .

It follows that T is a nice tile. □

Example 5.4 Here we show an example of a nice tile T , $|T| = 12$, of an $m \times n$ array with $m = 5$ and $n \geq 6$, that has 3 non-empty rows and 4 non-empty columns. The cells that belong to T are represented by a \bullet .

•	•		...		•	•
•	•		...		•	•
			...			
			...			
•	•		...		•	•

Proposition 5.5 *Let T be a set of cells of an $m \times n$ array that satisfies the following conditions:*

- 1) T has exactly 2 non-empty rows $R_\alpha, R_{\alpha+1}$ where the indices are considered modulo m ;
- 2) Each row $R_\alpha, R_{\alpha+1}$ is non-empty exactly in the cells of the $b \geq 4$ columns $C_\beta, C_{\beta+1}, \dots, C_{\beta+(b-1)}$ where the indices are considered modulo n .

Then T is nice.

Proof. Let us consider a set S of $|T| = 2b$ elements of a finite group G and vectors $(r_1, \dots, r_m) \in G^m, (c_1, \dots, c_n) \in G^n$. We proceed as in the proof of Proposition 5.3 by choosing, uniformly at random, an array A such that $\text{skel}(A) = T$ and $\mathcal{E}(A) = S$.

Here, denoting by $\mathbb{E}(X)$ the expected value of the random variable X given by the number of rows R_i , with $i \in \{\alpha, \alpha + 1\}$, that sums to r_i , we have that:

$$\mathbb{E}(X) \leq \frac{2}{2b - (b - 1)} = \frac{2}{b + 1}.$$

Similarly, if we denote by $\mathbb{E}(Y)$ the expected value of the random variable Y given by the number of columns C_j , with $j \in \{\beta, \beta + 1, \dots, \beta + (b - 1)\}$, that sums to c_j , we have that:

$$\mathbb{E}(Y) \leq \frac{b}{2b - 1}.$$

Since $b \geq 4$, we have that

$$\mathbb{E}(X) + \mathbb{E}(Y) \leq \frac{2}{b + 1} + \frac{b}{2b - 1} < 1.$$

Therefore there exists an array A whose skeleton is T and such that $\mathcal{E}(A) = S$ and:

- (d_1) if the i -th row of A is non-empty, the sum of its elements is different from r_i ;
- (d_2) if the j -th column of A is non-empty, the sum of its elements is different from c_j .

It follows that T is a nice tile. □

Example 5.6 Here we show an example of a nice tile $T, |T| = 10$, of an $m \times n$ array with $m = 3$ and $n \geq 7$, that has 2 non-empty rows and 5 non-empty columns.

•	•		⋯		•	•	•
			⋯				
•	•		⋯		•	•	•

Using the nice tiles of Propositions 5.3 and 5.5, we can prove that

Theorem 5.7 *Let G be a finite group and let J be a subgroup of G . Then there exists a ${}^\lambda\text{NH}_{G,J}(m, n)$ whenever the necessary conditions $(*)$ (and those of Theorem 2.3 in case G is isomorphic to \mathbb{Z}_2^r) are satisfied and $|G \setminus J| \geq 15$.*

Proof. We denote by B the set of cells of an $m \times n$ totally filled array and we assume, without loss of generality, that $m \leq n$.

In the following, we will assume that $m > 1$ since the case $m = 1$ has already been considered in Proposition 5.1. Since $m > 1$, we can partition the rows of B into sets $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ of consecutive rows such that each \mathcal{H}_i contains either two or three rows. Here we will say that \mathcal{H}_i has weight, respectively 2 or 3.

Now, if $n \geq 4$, we can partition the cells that belong to each set \mathcal{H}_i of weight 2 with tiles defined in Proposition 5.5: it suffices to use tiles with $b \in \{4, 5, 6, 7\}$ (i.e. that has 4, 5, 6 or 7 non-empty consecutive columns).

Similarly, if $n \geq 3$, we can partition the cells that belong to each set \mathcal{H}_i of weight 3 with tiles defined in Proposition 5.3: it suffices to use tiles with $b \in \{3, 4, 5\}$ (i.e. that has 3, 4 or 5 non-empty consecutive columns).

Summing up, if $n \geq 4$, we have that:

- 1) it is possible to partition B into nice tiles T_1, \dots, T_ℓ ;
- 2) $\max_i(|T_i|) \leq 15$.

Therefore, because of Theorem 4.4, there exists a ${}^\lambda\text{NH}_{G,J}(m, n)$ whenever the necessary conditions are satisfied, $n \geq 4$ and $|G \setminus J| \geq 15$. It follows that the thesis is proved whenever $n \geq 4$. Now we assume that $n < 4$ which implies that either $n = m = 3$ or $mn \leq 6$. In the first case, if $n = m = 3$, B itself is nice, due to Proposition 5.3. Therefore, because of Theorem 4.4, there exists a ${}^\lambda\text{NH}_{G,J}(3, 3)$ whenever the necessary conditions are satisfied and $|G \setminus J| \geq 15 > 9$.

Finally, let us suppose that $mn \leq 6$; we have that $|G \setminus J| \leq 12$ which is in contradiction with the hypothesis that $|G \setminus J| \geq 15$. It follows that there exists a ${}^\lambda\text{NH}_{G,J}(m, n)$ whenever the necessary conditions are satisfied and $|G \setminus J| \geq 15$. □

We remark that, for any group G whose order v is odd we have that $t \leq v/3$ while, if the order is even $t \leq v/2$ where t is the order of a subgroup J . It follows that

Corollary 5.8 *Let G be a group of size v and let J be a subgroup of G . Then there exists a ${}^\lambda\text{NH}_{G,J}(m, n)$ whenever the necessary conditions $(*)$ are satisfied (and those of Theorem 2.3 in case G is isomorphic to \mathbb{Z}_2^r) and $v \geq 30$ if v is even, $v \geq 23$ otherwise.*

6 Partially Filled Arrays

In this section, we will consider the case of arrays that are not totally filled. Since a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ is totally filled if and only if $n = h$, when needed, we will assume that $n > h$. Due to the necessary conditions (*), if a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ exists, we must have that $nk = mh$ that implies that $\text{lcm}(m, n) | (nk)$. In the following we will provide a construction of a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ according to whether $r = \frac{nk}{\text{lcm}(m, n)}$ is 1, 2 or at least 3.

6.1 Case $r = 1$

Proposition 6.1 *Let G be a group and let J be a subgroup of G . If $n > h = 1$, there exists a ${}^\lambda\text{NH}_{G,J}(m, n; 1, k)$ assuming that the necessary conditions (*) are satisfied and $|G \setminus J| \geq 4$.*

Proof. Note that due to the necessary condition $mh = nk$ and $h = 1$ we have that $n = m/k$. Now we denote by Q the set of cells given by

$$Q := \{(i, 1) : 1 \leq i \leq k\}.$$

We consider the subset of the $m \times n$ array defined by:

$$B := \bigcup_{i=0}^{m/k-1} Q + i(k, 1).$$

We order the elements of $G \setminus J$ as g_1, \dots, g_u in such a way that $g_i + g_{i+1} \neq 0$ for any $i \in \{1, \dots, u\}$ where the sum is considered modulo u . Note that this is possible since $u \geq 4$. Moreover, if λ is odd, we have that $G \setminus J$ does not contain any involution and thus we can assume that $\pm\{g_1, \dots, g_{\frac{u}{2}}\} = G \setminus J$ as done in the proof of Theorem 4.4.

Now we fill the elements of B so that, in the i -th row we put the element g_i where the index i is considered modulo u . We denote by A_1 the array so defined. Since the necessary conditions are satisfied, we have filled B with the elements of:

$$\Omega := \begin{cases} (G \setminus J)^{\lambda/2} & \text{if } \lambda \equiv 0 \pmod{2}; \\ (G \setminus J)^{(\lambda-1)/2} \cup \{g_1, \dots, g_{\frac{u}{2}}\} & \text{otherwise.} \end{cases}$$

In both cases, we have that $\pm\mathcal{E}(A_1) = \pm\Omega$ cover $G \setminus J$ exactly λ times. We note that the filled cells in the i -th column are those of $Q + (i - 1)(k, 1)$. Here, if $k = 1$ or $k = 2$ we obtain a ${}^\lambda\text{NH}_{G,J}(m, n; 1, k)$ since, for any $j \in \{1, \dots, u\}$, both g_j and $g_j + g_{j+1}$ are nonzero.

Let now assume $k \geq 3$ and $n \geq 2$ since $n = 1$ (or, symmetrically, $m = 1$) is excluded by the hypotheses and has already been considered in Proposition 5.1. We consider the first column C_i , whose elements are $g_j, g_{j+1}, \dots, g_{j+k-1}$ where $j = (i - 1)k$, that sums to zero. Here the next column is C_{i+1} whose elements are

$g_{j+k}, g_{j+k+1}, \dots, g_{j+2k-1}$. We note that, since $u \geq 4$, $g_{j+k-1} \notin \{g_{j+k}, g_{j+k+1}\}$ and hence given $x \in \{g_{j+k}, g_{j+k+1}\}$ we have that

$$g_j + g_{j+1} + \dots + g_{j+k-2} + x \neq 0.$$

Let now assume that

$$g_{j+k-1} + g_{j+k+1} + g_{j+k+2} + \dots + g_{j+2k-1} = 0, \tag{6.1}$$

since otherwise we can interchange the last element of C_i and the first of C_{i+1} to ensure that both these columns do not sum to zero. From equation (6.1) it follows that

$$g_{j+k-1} + g_{j+k} + g_{j+k+2} + \dots + g_{j+2k-1} \neq 0.$$

Therefore, also in this case, we can ensure that the columns C_i and C_{i+1} do not sum to zero by permuting the elements $\{g_{j+k-1}, g_{j+k}, g_{j+k+1}\}$ following the cycle $(g_{j+k-1}, g_{j+k}, g_{j+k+1})$.

Note that this procedure can be reiterated since it involves the last element of C_i , the first two elements of C_{i+1} , and since $k \geq 3$. Hence, we eventually obtain a ${}^\lambda\text{NH}_{G,J}(m, n; 1, k)$ also when $k \geq 3$. \square

Definition 6.2 We name by Q the set of cells of an $m \times n$ array given by

$$Q := \{(i, j) : 1 \leq i \leq k; 1 \leq j \leq h\}.$$

Then, given b such that $bk \leq m$ and $bh \leq n$, the set of cells given by

$$T := \bigcup_{i=0}^{b-1} Q + i(k, h)$$

is said to be an (k, h) -stair of length b .

Proposition 6.3 A $(3, 2)$ -stair of length b , T , is nice whenever $b \geq 2$.

Proof. Let us consider a set S of $|T| = 6b$ elements of a group G and vectors $(r_1, \dots, r_m) \in G^m$, $(c_1, \dots, c_n) \in G^n$. We note that T has exactly $3b$ rows each with 2 non-empty cells. Here we consider a non-empty row R_i and we denote by $(i, \beta), (i, \beta + 1)$ its non-empty cells where the sum is considered modulo n . Now we chose, uniformly at random, an array A such that $\text{skel}(A) = T$ and $\mathcal{E}(A) = S$ and we denote by $\mathbb{P}(X_i)$ the probability of the event X_i that the i -th row sums, following the natural ordering, to r_i . Here we have that, chosen the first element $a_{i,\beta}$ of the i -th row there is at most one element $x \in S$ such that

$$a_{i,\beta} + x = r_i.$$

It follows that

$$\mathbb{P}(X_i) \leq \frac{1}{|S| - 1} = \frac{1}{6b - 1}.$$

Now we denote by $\mathbb{E}(X)$ the expected value of the random variable X given by the number of non-empty rows R_i that sums to r_i . Due to the linearity of the expected value, we have that:

$$\mathbb{E}(X) = \sum_{i: \text{skel}(R_i) \cap T \neq \emptyset} \mathbb{P}(X_i) \leq \frac{3b}{6b - 1}.$$

Similarly, given a non-empty column C_j , we denote by $\mathbb{P}(Y_j)$ the probability of the event Y_j that C_j sums to c_j . Proceeding as we did for the rows, and noting that the number of the elements in the j -th column is three, we have that

$$\mathbb{P}(Y_i) \leq \frac{1}{|S| - 2} = \frac{1}{6b - 2}.$$

Therefore, using again the linearity of the expected value, and denoting by $\mathbb{E}(Y)$ the expected value of the random variable Y given by the number of non-empty columns C_j that sums to c_j , we have that:

$$\mathbb{E}(Y) = \sum_{j: \text{skel}(C_j) \cap T \neq \emptyset} \mathbb{P}(X_j) \leq \frac{2b}{6b - 2}.$$

It follows that

$$\mathbb{E}(X) + \mathbb{E}(Y) \leq \frac{3b}{6b - 1} + \frac{2b}{6b - 2} < \frac{5b}{6b - 2}.$$

Since we are assuming that $b \geq 2$, it follows that

$$\frac{5b}{6b - 2} \leq 1.$$

Therefore, there exists an assignment of the values of the cells of T among S that satisfies the conditions of Definition 4.3. □

Example 6.4 Here we show an example of a nice $(3, 2)$ -stair of length 3, T , $|T| = 18$, of an $m \times n$ array with $m = 9$ and $n = 6$.

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Now, as an application of the previous proposition, we use this tile in order to obtain a ${}^2\text{NH}_{G,J}(9, 6; 2, 3)$. More precisely, let $G = \mathbb{Z}_{36}$, $J = 2\mathbb{Z}_{36}$ and let $S = G \setminus J$ which has size $18 = |T|$. We fix the vectors $(r_1, \dots, r_9) = (0, \dots, 0)$ and $(c_1, \dots, c_6) = (0, \dots, 0)$. Since, due to Proposition 6.3 the tile T is nice, we obtain the existence of a ${}^2\text{NH}_{G,J}(9, 6; 2, 3)$ array A such that $\text{skel}(A) = T$ and $\mathcal{E}(A) = S$. We can consider, for instance

$$A := \begin{array}{|c|c|c|c|c|c|} \hline 1 & -3 & & & & \\ \hline 3 & -1 & & & & \\ \hline 5 & -7 & & & & \\ \hline & & 7 & -5 & & \\ \hline & & 9 & -11 & & \\ \hline & & 11 & -9 & & \\ \hline & & & & 13 & -15 \\ \hline & & & & 15 & -17 \\ \hline & & & & 17 & -13 \\ \hline \end{array}$$

Note that Proposition 6.3 can be used also to avoid other row and column sums. Here we consider again G to be \mathbb{Z}_{36} but now we set S to be $\{1, 2, \dots, 9\} \cup \{-1, -2, \dots, -9\}$ which has also size $18 = |T|$ and we fix the vectors $(r_1, \dots, r_9) = (1, \dots, 1)$ and $(c_1, \dots, c_6) = (6, \dots, 6)$. Since by Proposition 6.3 we know that T is a nice tile, there exists an array A such that $\text{skel}(A) = T$, $\mathcal{E}(A) = S$ and where the sum of the elements in the i -th row (respectively i -th column) is different from the value r_i (respectively c_i). We can consider, for instance

$$A := \begin{array}{|c|c|c|c|c|c|} \hline 1 & -1 & & & & \\ \hline 2 & -2 & & & & \\ \hline 4 & -4 & & & & \\ \hline & & 3 & -3 & & \\ \hline & & 5 & -5 & & \\ \hline & & 6 & -6 & & \\ \hline & & & & 7 & -7 \\ \hline & & & & 8 & -8 \\ \hline & & & & 9 & -9 \\ \hline \end{array}$$

Proposition 6.5 *Let G be a group and let J be a subgroup of G . If $n > h$, there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ assuming that $r = \frac{nk}{\text{lcm}(m,n)} = 1$, the necessary conditions (*) are satisfied and $|G \setminus J| \geq 18$.*

Proof. In this case, we have $nk = mh = \text{lcm}(m, n)$. It follows that

$$k = \frac{\text{lcm}(m, n)}{n}$$

and

$$h = \frac{\text{lcm}(m, n)}{m}.$$

We name by Q the set of cells given by

$$Q := \{(i, j) : 1 \leq i \leq k; 1 \leq j \leq h\}.$$

Then, since

$$\frac{n}{h} = \frac{m}{k} = \frac{mn}{\text{lcm}(m, n)} = \text{gcd}(m, n),$$

we can consider the subset of the $m \times n$ array defined by:

$$B := \bigcup_{i=0}^{n/h-1} Q + i(k, h).$$

Note that $k = h$ would imply that $m = n = \text{lcm}(m, n)$ and hence, since, $r = 1$ it would be possible only if $k = h = 1$. Let $u := |G \setminus J|$. In this case any array A such that $\text{skel}(A) = B$ and $\mathcal{E}(A)$ is Ω where

$$\Omega := \begin{cases} (G \setminus J)^{\lambda/2} & \text{if } \lambda \equiv 0 \pmod{2}; \\ (G \setminus J)^{(\lambda-1)/2} \cup \{g_1, \dots, g_{\frac{u}{2}}\} & \text{otherwise,} \end{cases}$$

is a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$. Here, if λ is odd, we are assuming that $\pm\{g_1, \dots, g_{\frac{u}{2}}\} = G \setminus J$, as done in the proof of Theorem 4.4.

We can suppose now that $h < k$.

CASE $h = 1$: This case follows from Proposition 6.1.

CASE $h = 2$ and $k = 3$: Here we can assume that $(m, n) \neq (3, 2)$ since otherwise B would be the skeleton of a totally filled array. In this case, B can be tessellated with tiles of type $(3, 2)$ -stair of length either two or three. Since those tiles are nice and their sizes are at most 18, they define a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$.

Otherwise we have that $h \geq 3$ and $k > h$. In this case, we have that Q itself can be tessellated with tiles whose sizes are at most 15 as done in Theorem 5.7. Hence the same can be done, by translation, with B .

It follows that, assuming $r = 1$, there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ whenever the necessary conditions are satisfied and $|G \setminus J| \geq 18$. □

6.2 Case $r = 2$

Proposition 6.6 *Let $(G, +)$ be a finite group and let J be a subgroup of G . Then there exists a ${}^\lambda\text{NH}_{G,J}(n; 2)$ assuming that the necessary conditions $(*)$ are satisfied and $|G \setminus J| \geq 4$.*

Proof. We name by Q the set of cells given by

$$Q := \{(1, j) : 1 \leq j \leq 2\}.$$

We consider the subset of the $n \times n$ array defined by:

$$B := \bigcup_{i=0}^{n-1} Q + i(1, 1)$$

where the sum is considered modulo n .

We order the elements of $G \setminus J$ as g_1, \dots, g_u in such a way that $g_i + g_{i+1} \neq 0$ for any $i \in \{1, \dots, u\}$ where the sum is considered modulo u . Note that this is possible since $u \geq 4$. Moreover, if λ is odd, we have that $G \setminus J$ does not contain any involution and thus, as done in the proof of Theorem 4.4, we can assume that $\pm\{g_1, \dots, g_{\frac{u}{2}}\} = G \setminus J$.

Now we divide the proof according to whether λ is even or odd.

CASE 1: λ is even. Here we fill the elements of B so that, we put in the i -th row, in order, the elements g_{2i-1} and g_{2i} where the indices are considered modulo u . We denote by A the array so defined.

Since the necessary conditions are satisfied, we have that $\text{skel}(A) = B$ and $\mathcal{E}(A) = \Omega$ where

$$\Omega = (G \setminus J)^{\lambda/2}.$$

Moreover, because of the definition and considering the indices modulo u , we have that for any $i \in [1, n]$,

$$\begin{cases} g_{2i-1} + g_{2i} \neq 0; \\ g_{2i} + g_{2i+1} \neq 0. \end{cases}$$

Note that, since λ is even, here we have that $\mathcal{E}(C_1) = \{g_1, g_u\}$. It follows that $\mathcal{E}(R_i) = \{g_{2i-1}, g_{2i}\}$ and $\mathcal{E}(C_{i+1}) = \{g_{2i}, g_{2i+1}\}$, where the row indices are considered modulo n and the group ones modulo u , and hence A is a ${}^\lambda\text{NH}_{G,J}(n; 2)$.

CASE 2: λ is odd. Here we fill the elements of B so that, in the i -th row we put, in order, the elements g_{2i-1} and g_{2i} where the indices are considered modulo $\frac{u}{2}$. We denote by A the array so defined.

Since the necessary conditions are satisfied, we have that $\text{skel}(A) = B$ and $\mathcal{E}(A) = \Omega$ where

$$\Omega = \{g_1, \dots, g_{\frac{u}{2}}\}^\lambda.$$

Moreover, because of the definition of A , we have that if $x \in \mathcal{E}(A)$, then $-x \notin \mathcal{E}(A)$. It follows that the sum over each row and column is non-zero. Therefore, A is a ${}^\lambda\text{NH}_{G,J}(n; 2)$. □

Definition 6.7 Let h, k be positive even integers. We name by Q the set of cells of an $m \times n$ array given by

$$Q := \{(i, j) : 1 \leq i \leq k/2; 1 \leq j \leq h/2\}.$$

Then, given b such that $bk/2 \leq m$ and $bh/2 \leq n$, the set of cells given by

$$T := \left(\bigcup_{i=0}^{b-1} Q + i(k/2, h/2) \right) \cup \left(\bigcup_{i=0}^{b-1} Q + (0, h/2) + i(k/2, h/2) \right)$$

is said to be a double $(k/2, h/2)$ -stair of length b .

Proposition 6.8 *A double (2, 1)-stair of length b, T, is nice whenever b ≥ 3.*

Proof. Let us consider a set S of |T| = 4b elements of a finite group (G, +), and vectors (r₁, . . . , r_m) ∈ G^m, (c₁, . . . , c_n) ∈ Gⁿ. We note that T has exactly 2b rows each with 2 non-empty cells. Here we consider a non-empty row R_i and we denote by (i, β), (i, β + 1) its non-empty cells where the sum is considered modulo n. Now we chose, uniformly at random, an array A such that skel(A) = T and E(A) = S and we denote by P(X_i) the probability of the event X_i that the i-th row sums, following the natural ordering, to r_i. Here we have that, chosen the first element a_{i,β} of the i-th row there is at most one element x ∈ S such that

$$a_{i,\beta} + x = r_i.$$

It follows that

$$P(X_i) \leq \frac{1}{|S| - 1} = \frac{1}{4b - 1}.$$

Now we denote by E(X) the expected value of the random variable X given by the number of non-empty rows R_i that sums to r_i. Due to the linearity of the expected value, we have that:

$$E(X) = \sum_{i: \text{skel}(R_i) \cap T \neq \emptyset} P(X_i) \leq \frac{2b}{4b - 1}.$$

Similarly, given a non-empty column C_j, we denote by P(Y_j) the probability of the event Y_j that C_j sums to c_j. Proceeding as we did for the rows, and noting that the number of the elements of the j-th column is four except when j = 1 or j = b + 1 when is two, we have that, if j ∈ {2, 3, . . . , b}

$$P(Y_j) \leq \frac{1}{|S| - 3} = \frac{1}{4b - 3}$$

and

$$P(Y_1) = P(Y_{b+1}) \leq \frac{1}{|S| - 1} = \frac{1}{4b - 1}.$$

Therefore, using again the linearity of the expected value, and denoting by E(Y) the expected value of the random variable Y given by the number of non-empty columns C_j that sums to c_j, we have that:

$$E(Y) = \sum_{j: \text{skel}(C_j) \cap T \neq \emptyset} P(Y_j) \leq \frac{b - 1}{4b - 3} + \frac{2}{4b - 1}.$$

It follows that, assuming b ≥ 3,

$$E(X) + E(Y) \leq \frac{2b}{4b - 1} + \frac{b - 1}{4b - 3} + \frac{2}{4b - 1} < 1.$$

Therefore, there exists an assignment of the values of the cells of T among S that satisfies the conditions of Definition 4.3. □

Example 6.9 Here we show an example of a double (2, 1)-stair of length 3, T , $|T| = 12$, of an $m \times n$ array with $m = 6$ and $n = 4$.

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Now we consider the group $G = \mathbb{Z}_{24}$; we set S to be $\{1, 2, \dots, 6\} \cup \{-1, -2, \dots, -6\}$ which has size $12 = |T|$, and we fix the vectors $(r_1, \dots, r_6) = (1, \dots, 1)$ and $(c_1, c_2, c_3, c_4) = (3, 8, -2, 1)$. Since by Proposition 6.8 we know that T is a nice tile, there exists an array A such that $\text{skel}(A) = T$, $\mathcal{E}(A) = S$ and where the sum of the elements in the i -th row (respectively i -th column) is different from the value r_i (respectively c_i). We can consider, for instance

$$A := \begin{array}{|c|c|c|c|} \hline 1 & -1 & & \\ \hline -2 & 2 & & \\ \hline & -3 & 3 & \\ \hline & 4 & -4 & \\ \hline & & 5 & -5 \\ \hline & & 6 & -6 \\ \hline \end{array}$$

With essentially the same proof we also obtain that:

Proposition 6.10 *A double (3, 1)-stair of length b , T , is nice whenever $b \geq 2$.*

Example 6.11 Here we show an example of a double (3, 1)-stair of length 3, T , $|T| = 18$, of an $m \times n$ array with $m = 9$ and $n = 4$.

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Now we consider the group $G = \mathbb{Z}_{36}$; we set S to be $\{1, 2, \dots, 9\} \cup \{-1, -2, \dots, -9\}$ which has size $18 = |T|$, and we fix the vectors $(r_1, \dots, r_9) = (1, \dots, 1)$ and $(c_1, c_2, c_3, c_4) = (3, 4, 3, 0)$. Since by Proposition 6.10 we know that T is a nice tile, there exists an array A such that $\text{skel}(A) = T$, $\mathcal{E}(A) = S$ and where the sum of the elements in the i -th row (respectively i -th column) is different from the value r_i (respectively c_i). We can consider, for instance

$$A := \begin{array}{|c|c|c|c|} \hline 1 & -1 & & \\ \hline -2 & 2 & & \\ \hline 3 & -3 & & \\ \hline & 4 & -4 & \\ \hline & -5 & 5 & \\ \hline & 6 & -6 & \\ \hline & & 7 & -7 \\ \hline & & -8 & 8 \\ \hline & & -9 & 9 \\ \hline \end{array}$$

We are now ready to prove the following result.

Proposition 6.12 *Let $(G, +)$ be a finite group and let J be a subgroup of G . Then there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ assuming that $r = \frac{nk}{\text{lcm}(m,n)} = 2$, the necessary conditions $(*)$ are satisfied and $|G \setminus J| \geq 20$.*

Proof. In this case, we have that $nk = mh = 2 \text{lcm}(m, n)$. It follows that both k, h are even,

$$\frac{k}{2} = \frac{\text{lcm}(m, n)}{n}$$

and

$$\frac{h}{2} = \frac{\text{lcm}(m, n)}{m}.$$

We name by Q the set of cells given by

$$Q := \{(i, j) : 1 \leq i \leq k/2; 1 \leq j \leq h/2\}.$$

Then, since

$$\frac{2n}{h} = \frac{2m}{k} = \frac{mn}{\text{lcm}(m, n)} = \text{gcd}(m, n),$$

we can consider the subset of the $m \times n$ array defined by:

$$B := \left(\bigcup_{i=0}^{2n/h-1} Q + i(k/2, h/2) \right) \cup \left(\bigcup_{i=0}^{2n/h-1} Q + (0, h/2) + i(k/2, h/2) \right).$$

Note that $k = h$ would imply that $m = n = \text{lcm}(m, n)$ and hence, since, $r = 2$ it would be possible only if $k = h = 2$. In this case, the thesis follows from Proposition 6.6.

We can suppose now that $h < k$.

CASE $h = 2$ and $k = 4$: In this case, m is at least 6 and B can be tessellated with double $(2, 1)$ -stair of length either three or four or five. Since those tiles are nice and their sizes are at most 20, they define a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$.

CASE $h = 2$ and $k = 6$: In this case, m is at least 9 and B can be tessellated with double $(3, 1)$ -stair of length either two or three. Since those tiles are nice and their sizes are at most 18, they define a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$.

CASE $h = 2$ and $k \geq 8$: In this case $Q \cup (Q + (0, h/2))$ is a $(k/2) \times 2$ rectangle and, since $k/2 \geq 4$, can be tessellated with tiles whose sizes are at most 15 as done in Theorem 5.7. Hence the same can be done, by translation, with B . It follows that, also in this case, we obtain a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$.

Otherwise we have that $h \geq 4$ and $k > h$. In this case $Q \cup (Q + (0, h/2))$ is a $(k/2) \times h$ rectangle and, since $k/2 \geq 3$ and $h \geq 4$, can be tessellated with tiles whose sizes are at most 15, again as done in the proof of Theorem 5.7. Hence the same can be done, by translation, with B .

It follows that, assuming $r = 2$, there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ whenever the necessary conditions are satisfied and $|G \setminus J| \geq 20$. □

Example 6.13 Let $G = \text{Alt}(5)$ be the alternating group of degree 5, i.e., the subgroup of order 60 given by the even permutations of the symmetric group S_5 . Set $J = \langle (1, 2)(3, 4), (1, 2, 3) \rangle$, then it can be seen that $|G \setminus J| = 48$.

We provide an example of a ${}^2\text{NH}_{G,J}(12, 8; 4, 6)$ in Figure 1 that is constructed using Proposition 6.12. Since $r = \frac{nk}{\text{lcm}(m,n)} = 2$ and $|G \setminus J| = 48 \geq 20$, the hypotheses of Proposition 6.12 are satisfied and hence we can use four of the 3×4 nice tiles defined in Proposition 5.3 to tessellate the skeleton of the array (the colors in Figure 1 represent the nice tiles used to construct the array).

Let A be the array shown in Figure 1; then it is easy to see that $\mathcal{E}(A) = G \setminus J = [(3, 4, 5), (3, 5, 4), (2, 3)(4, 5), (2, 3, 5), (2, 4, 5), (2, 4)(3, 5), (2, 5, 3), (2, 5, 4), (2, 5)(3, 4), (1, 2)(4, 5), (1, 2)(3, 5), (1, 2, 3, 4, 5), (1, 2, 3, 5, 4), (1, 2, 4, 5, 3), (1, 2, 4, 3, 5), (1, 2, 5, 4, 3), (1, 2, 5), (1, 2, 5, 3, 4), (1, 3, 4, 5, 2), (1, 3, 5, 4, 2), (1, 3)(4, 5), (1, 3, 5), (1, 3, 2, 4, 5), (1, 3, 5, 2, 4), (1, 3)(2, 5), (1, 3, 2, 5, 4), (1, 3, 4, 2, 5), (1, 4, 5, 3, 2), (1, 4, 3, 5, 2), (1, 4, 5), (1, 4)(3, 5), (1, 4, 5, 2, 3), (1, 4, 2, 3, 5), (1, 4, 2, 5, 3), (1, 4, 3, 2, 5), (1, 4)(2, 5), (1, 5, 3, 4, 2), (1, 5, 3), (1, 5, 4, 3, 2), (1, 5, 2), (1, 5)(2, 3), (1, 5, 4, 2, 3), (1, 5, 4), (1, 5)(3, 4), (1, 5, 3, 2, 4), (1, 5)(2, 4), (1, 5, 2, 3, 4), (1, 5, 2, 4, 3)]$, and hence $[\pm x \mid x \in \mathcal{E}(A)]$ covers $G \setminus J$ twice.

6.3 The general case $r \geq 3$

In this case, we consider the following kind of tiles.

Definition 6.14 Let m, n be two positive integers such that $n \geq m$. Identified the $m \times n$ array with the elements of $\mathbb{Z}_m \oplus \mathbb{Z}_n$, and given $a < n, b \leq m$, we say that a set T of cells is an (a, b) -diagonal tile if its non-empty cells are exactly the following ones:

$$\{(\bar{i}, \bar{j}) + (x, x), (\bar{i}, \bar{j} + 1) + (x, x), \dots, (\bar{i}, \bar{j} + (a - 1)) + (x, x) : x \in [0, b - 1]\}$$

for some $(\bar{i}, \bar{j}) \in \mathbb{Z}_m \oplus \mathbb{Z}_n$.

Proposition 6.15 A $(3, b)$ -diagonal tile T is nice whenever $b \geq 4$ and either $b \leq n - 2$ or $b = m$.

(3, 4, 5)	(3, 5, 4)	(2, 3)(4, 5)	(2, 3, 5)					
(2, 4, 5)	(2, 4)(3, 5)	(2, 5, 3)	(2, 5, 4)					
(2, 5)(3, 4)	(1, 2)(4, 5)	(1, 2)(3, 5)	(1, 2, 3, 4, 5)					
		(1, 2, 3, 5, 4)	(1, 2, 4, 5, 3)	(1, 2, 4, 3, 5)	(1, 2, 5, 4, 3)			
		(1, 2, 5)	(1, 2, 5, 3, 4)	(1, 3, 4, 5, 2)	(1, 3, 5, 4, 2)			
		(1, 3)(4, 5)	(1, 3, 5)	(1, 3, 2, 4, 5)	(1, 3, 5, 2, 4)			
				(1, 3)(2, 5)	(1, 3, 2, 5, 4)	(1, 3, 4, 2, 5)	(1, 4, 5, 3, 2)	
				(1, 4, 3, 5, 2)	(1, 4, 5)	(1, 4)(3, 5)	(1, 4, 5, 2, 3)	
				(1, 4, 2, 3, 5)	(1, 4, 2, 5, 3)	(1, 4, 3, 2, 5)	(1, 4)(2, 5)	
(1, 5, 3, 4, 2)	(1, 5, 3)					(1, 5, 4, 3, 2)	(1, 5, 2)	
(1, 5)(2, 3)	(1, 5, 4, 2, 3)					(1, 5, 4)	(1, 5)(3, 4)	
(1, 5, 3, 2, 4)	(1, 5)(2, 4)					(1, 5, 2, 3, 4)	(1, 5, 2, 4, 3)	

Figure 1: $A^2\text{NH}_{G,J}(12, 8; 4, 6)$ where $G = \text{Alt}(5)$ is the alternating group of order 5 and $J = \langle (1, 2)(3, 4), (1, 2, 3) \rangle$.

Proof. First, we note that the property (c_1) of Definition 4.3 is always satisfied while property (c_2) holds since either $b \leq n - 2$ or $b = m$.

Let us consider a set S of $|T| = 3b$ elements of a group G and vectors $(r_1, \dots, r_m) \in G^m$, $(c_1, \dots, c_n) \in G^n$. Since T is a $(3, b)$ -diagonal tile, it has exactly b rows each with three non-empty cells. We consider a non-empty row R_i and we denote by $(i, \beta), (i, \beta + 1), (i, \beta + 2)$ its non-empty cells where the sum is considered modulo n . Now we choose, uniformly at random, an array A such that $\text{skel}(A) = T$ and $\mathcal{E}(A) = S$, and we denote by $\mathbb{P}(X_i)$ the probability of the event X_i that the i -th row sums, following the natural ordering, to r_i . Here we have that, having chosen the first two elements $a_{i,\beta}, a_{i,\beta+1}$ of the i -th row, there is at most one element $x \in S$ such that $a_{i,\beta} + a_{i,\beta+1} + x = r_i$. It follows that

$$\mathbb{P}(X_i) \leq \frac{1}{|S| - 2} = \frac{1}{3b - 2}.$$

Now we denote by $\mathbb{E}(X)$ the expected value of the random variable X given by the number of non-empty rows R_i that sum to r_i . Due to the linearity of the expected value, we have that:

$$\mathbb{E}(X) = \sum_{i: \text{skel}(R_i) \cap T \neq \emptyset} \mathbb{P}(X_i) \leq \frac{b}{3b - 2}.$$

Similarly, given a non-empty column C_j , we denote by $\mathbb{P}(Y_j)$ the probability of the event Y_j that C_j sums to c_j . Proceeding as we did for the rows, denoting by $|C_j|$ the number of the elements of the j -th column, we have

$$\mathbb{P}(Y_j) \leq \frac{1}{|S| - |C_j| + 1} = \frac{1}{3b - |C_j| + 1}.$$

To estimate the expected value $\mathbb{E}(Y)$ of the random variable Y given by the number of non-empty columns C_j that sums to c_j , we divide the discussion into three cases.

CASE 1: $b \leq n - 2$. Here T has exactly $b - 2$ columns with 3 non-empty cells, two columns with 2 non-empty cells, and two columns with one non-empty cell. Due to the linearity of the expected value, we have:

$$\mathbb{E}(Y) \leq \frac{b - 2}{3b - 2} + \frac{2}{3b} + \frac{2}{3b - 1}.$$

It follows that

$$\mathbb{E}(X) + \mathbb{E}(Y) \leq \frac{b}{3b - 2} + \frac{b - 2}{3b - 2} + \frac{2}{3b} + \frac{2}{3b - 1} < 1.$$

CASE 2: $b = n - 1$. Here T has exactly $b - 1$ columns with 3 non-empty cells and two columns with 2 non-empty cells. Due to the linearity of the expected value, we have:

$$\mathbb{E}(Y) \leq \frac{b - 1}{3b - 2} + \frac{2}{3b - 1}.$$

It follows that

$$\mathbb{E}(X) + \mathbb{E}(Y) \leq \frac{b}{3b-2} + \frac{b-1}{3b-2} + \frac{2}{3b-1} < 1.$$

CASE 3: $b = n$. Here T has exactly b columns with 3 non-empty cells. Due to the linearity of the expected value, we have:

$$\mathbb{E}(Y) \leq \frac{b}{3b-2}.$$

It follows that

$$\mathbb{E}(X) + \mathbb{E}(Y) \leq \frac{2b}{3b-2} < 1.$$

□

Example 6.16 Here we provide an example of a $(3, b)$ -diagonal tile T of an $m \times n$ array with $n \geq m = b = n - 2$.

•	•	•						
	•	•	•					
		•	•	•				
			•	•	•			
				⋮	⋮	⋮		
					•	•	•	
						•	•	•

Now we suppose that T is a $(3, 7)$ -diagonal tile of an $m \times n$ array with $m = 7$ and $n = 9$. Here we consider the group $G = \mathbb{Z}_{28}$, we set S to be $\{1, 2, \dots, 11\} \cup \{-1, -2, \dots, -10\}$ which has size $21 = |T|$ and we fix the vectors $(r_1, \dots, r_7) = (0, \dots, 0)$ and $(c_1, \dots, c_9) = (2, \dots, 2)$. Since by Proposition 6.15 we know that T is a nice tile, there exists an array A such that $\text{skel}(A) = T$, $\mathcal{E}(A) = S$ and where the sum of the elements in the i -th row (respectively i -th column) is different from the value r_i (respectively c_i). We can consider, for instance

$A :=$

1	-1	2						
	-2	3	-3					
		4	-4	5				
			-5	6	-6			
				7	-7	8		
					-8	9	-9	
						10	-10	11

With essentially the same proof we also have that:

Proposition 6.17 *A $(4, b)$ -diagonal tile T is nice whenever $b \geq 3$ and either $b \leq n - 3$ or $b = m$.*

Example 6.18 Here we provide an example of a $(4, b)$ -diagonal tile T of an $m \times n$ array with $n \geq m = b = n - 2$.

•	•	•	•						
	•	•	•	•					
		•	•	•	•				
			•	•	•	•			
				⋮	⋮	⋮	⋮		
					•	•	•	•	
						•	•	•	•
•							•	•	•

Proposition 6.19 A $(5, b)$ -diagonal tile T is nice whenever $b \geq 2$ and either $b \leq n - 4$ or $b = m$.

Example 6.20 Here we provide an example of a $(5, b)$ -diagonal tile T of an $m \times n$ array with $n \geq m = b = n - 1$.

•	•	•	•	•					
	•	•	•	•	•				
		•	•	•	•	•			
			⋮	⋮	⋮	⋮	⋮		
				•	•	•	•	•	
•					•	•	•	•	
•	•					•	•	•	
•	•	•					•	•	

Proposition 6.21 Let $(G, +)$ be a finite group and let J be a subgroup of G . Then there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ assuming that $r = \frac{nk}{\text{lcm}(m,n)} \geq 3$, the necessary conditions $(*)$ are satisfied and $|G \setminus J| \geq 21$.

Proof. We can assume, without loss of generality, that $n \geq m$ and that the array is not totally filled. Here we have that, necessarily, $n \geq m > r \geq 3$. Otherwise, since the array is not totally filled, we would have $k < 3$, and hence

$$r = \frac{nk}{\text{lcm}(m, n)} < \frac{3n}{\text{lcm}(m, n)} < 3,$$

contradicting the hypothesis that $r \geq 3$.

Assuming now that $n \geq m > 3$, $nk = mh$ and $r = \frac{nk}{\text{lcm}(m,n)} \geq 3$, we identify the cells of an $m \times n$ array with the elements of the group $\mathbb{Z}_m \oplus \mathbb{Z}_n$ and we consider the subgroup H generated by $(1, 1)$. It is easy to see that H contains exactly $\frac{\text{lcm}(m,n)}{m}$ filled cells in each row and $\frac{\text{lcm}(m,n)}{n}$ filled cells in each column. Moreover, since the

array is not totally filled, $H, H + (0, 1), \dots, H + (0, r - 1)$ are disjoint cosets. Now we set

$$B := \bigcup_{i=0}^{r-1} \{H + (0, i)\}.$$

Then B contains exactly

$$\frac{mh}{\text{lcm}(m, n)} \frac{\text{lcm}(m, n)}{m} = h$$

filled cells in each row and

$$\frac{nk}{\text{lcm}(m, n)} \frac{\text{lcm}(m, n)}{n} = k$$

filled cells in each column.

Since $r \geq 3$, we can partition the family of cosets $\mathcal{H} := \{H, H + (0, 1), \dots, H + (0, r - 1)\}$ into subfamilies $\mathcal{H}_1, \dots, \mathcal{H}_\ell$ of adjacent ones each of which contains (i.e. it has weight) either 3, 4 or 5 cosets. This means that each of those families is of the following form $\mathcal{H}_i = \{H + (0, j), H + (0, j + 1), \dots, H + (0, j + w - 1)\}$ where w is the weight of \mathcal{H}_i and $j \in [0, r - w]$.

Now, if $w = 3$ and since $m \geq 4$, the cells that belong to \mathcal{H}_i can be tessellated with $(3, b)$ -diagonal tiles each of which has $b \in \{4, 5, 6, 7\}$ and $b = m$ when $m \in \{4, 5, 6, 7\}$ or $b \leq m - 4 < n - 2$ otherwise. If instead $w = 4$ we have that $m > r \geq 4$. Then the cells that belong to \mathcal{H}_i can be tessellated with $(4, b)$ -diagonal tiles each of which has $b \in \{3, 4, 5\}$ and $b = m$ when $m = 5$ or $b \leq m - 3 \leq n - 3$ otherwise. Finally, if $w = 5$ we have that $m > r \geq 5$. Then the cells that belong to \mathcal{H}_i can be tessellated with $(5, b)$ -diagonal tiles each of which has $b \in \{2, 3\}$ and $b \leq m - 4 \leq n - 4$.

Since all those tiles are nice and their sizes are at most 21, we have that there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ assuming that $r = \frac{nk}{\text{lcm}(m, n)} \geq 3$, the necessary conditions are satisfied and $|G \setminus J| \geq 21$. □

Example 6.22 We provide an example in Figure 2 of the family of cosets \mathcal{H} , introduced in the previous proposition, for $m = n = 10, k = h = 6$, and $r = 6$. The black dots represent the cells contained in the set B , the green cells are the elements of the family $\mathcal{H}_1 = \{H, H + (0, 1), H + (0, 2)\}$ and the red cells are the elements of the family $\mathcal{H}_2 = \{H + (0, 3), H + (0, 4), H + (0, 5)\}$.

Now, we provide an example in Figure 3 where the cells that belong to \mathcal{H}_1 and \mathcal{H}_2 can be tessellated using four $(3, 5)$ -diagonal tiles.

From the result of this section, reasoning as in Corollary 5.8, we obtain the main result of this paper.

Theorem 6.23 *Let G be a group of size v and let J be a subgroup of G . If $n > h$, there exists a ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ assuming that the necessary conditions $(*)$ are satisfied and $v \geq 42$ if v is even, $v \geq 29$ otherwise.*

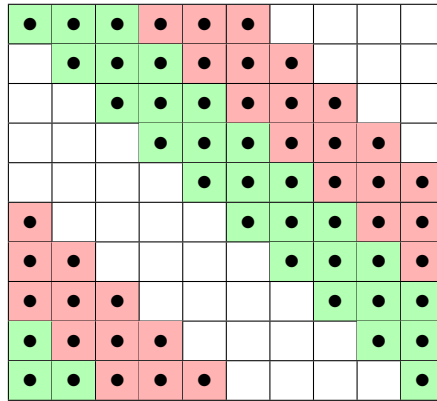


Figure 2: Example of the family of cosets \mathcal{H} , defined in Proposition 6.21, for $m = n = 10$, $k = h = 6$, and $r = 6$.

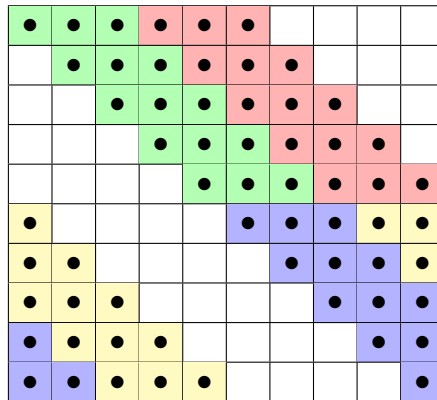


Figure 3: Example of a possible tessellation of the cells that belong to \mathcal{H}_1 and \mathcal{H}_2 using four $(3, 5)$ -diagonal tiles.

7 Application to Biembeddings

In [3], Archdeacon introduced Heffter arrays also because they are useful for finding biembeddings of cycle decompositions, as shown, for instance, in [8, 9, 12, 13, 14, 15, 16]. In this section, generalizing some of his results we show how starting from a λ -fold non-zero sum Heffter array it is possible to obtain suitable biembeddings. In this context, we equip a multigraph Γ with the following topology.

- If Γ is a simple graph, Γ is viewed with the usual topology as a 1-dimensional simplicial complex.
- If Γ is a multigraph, we consider the topology naturally induced on Γ by a simple graph Γ' that is a subdivision of Γ .

Note that this topology is well-defined because two multigraphs Γ and Γ' are homeomorphic if and only if there exists an isomorphism from some subdivision of Γ to

some subdivision of Γ' . Now we provide the following definition; see [19, 23, 24] for the simple graph case.

Definition 7.1 An *embedding* of a multigraph Γ in a surface Σ is a continuous injective mapping $\psi : \Gamma \rightarrow \Sigma$, where Γ is viewed with the topology described above.

The connected components of $\Sigma \setminus \psi(\Gamma)$ are called *ψ -faces*. If each ψ -face is homeomorphic to an open disc, then the embedding ψ is said to be *cellular*.

Definition 7.2 An embedding ψ of a multigraph Γ in a surface Σ is said to be a *biembedding* if it is face 2-colourable.

Given a λ -fold non-zero sum Heffter array ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$, say A , and permutations $\omega_{r_1}, \dots, \omega_{r_m}$ for the cells of each row, and $\omega_{c_1}, \dots, \omega_{c_n}$ for the cells of each column, we define $\omega_r := \omega_{r_1} \circ \omega_{r_2} \circ \dots \circ \omega_{r_m}$ and $\omega_c := \omega_{c_1} \circ \omega_{c_2} \circ \dots \circ \omega_{c_n}$. Then we say that ω_r and ω_c are *compatible* if $\omega_c \circ \omega_r$ is a cycle of length $|\text{skel}(A)|$.

Following the proof of Theorem 5.5 of [14], we obtain:

Theorem 7.3 Let $(G, +)$ be a finite group, J be a subgroup of G of order t and let A be a λ -fold relative Heffter array ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ that admits compatible orderings ω_r and ω_c . Then there exists a cellular biembedding of ${}^\lambda K_{\binom{2mk}{\lambda t} + 1} \times t$ into an orientable surface whose face lengths are multiples of k and h .

Moreover, if the sum of the elements in every row (following the ordering ω_r) and the sum of the elements in every column (following the ordering ω_c) are non-zero, the face lengths are strictly larger than k and h respectively.

Remark 7.4 Let A be a λ -fold relative Heffter array ${}^\lambda\text{NH}_{G,J}(m, n; h, k)$ as in the previous theorem and let G be abelian. In this case, the sum of the elements in a given row/column does not depend on the chosen ordering and is hence always non-zero. Here the existence of compatible orderings implies that there exists a cellular biembedding of ${}^\lambda K_{\binom{2mk}{\lambda t} + 1} \times t$ into an orientable surface whose face lengths are multiples of k and h strictly larger than k and h respectively.

As already remarked in [13], looking for compatible orderings led us to investigate the following problem introduced in [10]. Let A be an $m \times n$ toroidal p.f. array. By r_i we denote the orientation of the i -th row, precisely $r_i = 1$ if it is from left to right and $r_i = -1$ if it is from right to left. Analogously, for the j -th column, if its orientation c_j is from top to bottom then $c_j = 1$ otherwise $c_j = -1$. Assume that an orientation $\mathcal{R} = (r_1, \dots, r_m)$ and $\mathcal{C} = (c_1, \dots, c_n)$ is fixed. Given an initial filled cell (i_1, j_1) consider the sequence $L_{\mathcal{R},\mathcal{C}}(i_1, j_1) = ((i_1, j_1), (i_2, j_2), \dots, (i_\ell, j_\ell), (i_{\ell+1}, j_{\ell+1}), \dots)$ where $j_{\ell+1}$ is the column index of the filled cell $(i_\ell, j_{\ell+1})$ of the row R_{i_ℓ} next to (i_ℓ, j_ℓ) in the orientation r_{i_ℓ} , and where $i_{\ell+1}$ is the row index of the filled cell of the column $C_{j_{\ell+1}}$ next to $(i_\ell, j_{\ell+1})$ in the orientation $c_{j_{\ell+1}}$. The problem is the following:

Crazy Knight’s Tour Problem. Given a toroidal p.f. array A , do there exist \mathcal{R} and \mathcal{C} such that the list $L_{\mathcal{R},\mathcal{C}}$ covers all the filled cells of A ?

By $P(A)$ we will denote the Crazy Knight's Tour Problem for a given array A . Also, given a filled cell (i, j) , if $L_{\mathcal{R}, \mathcal{C}}(i, j)$ covers all the filled positions of A we will say that $(\mathcal{R}, \mathcal{C})$ is a solution of $P(A)$. For known results about this problem see [10]. The relationship between the Crazy Knight's Tour Problem and λ -fold non-zero sum Heffter arrays is explained in the following result which is an easy consequence of Theorem 7.3. For notation simplicity, we will state it only in the case of abelian groups.

Corollary 7.5 *Let $(G, +)$ be a finite abelian group, J be a subgroup of G of order t and let A be a λ -fold relative Heffter array ${}^\lambda\text{NH}_{G, J}(m, n; h, k)$ such that $P(A)$ admits a solution $(\mathcal{R}, \mathcal{C})$. Then there exists a cellular biembedding of ${}^\lambda K_{(\frac{2nk}{\lambda t} + 1) \times t}$ into an orientable surface whose face lengths are multiples of k and h strictly larger than k and h respectively.*

To present an existence result about biembeddings, we need to introduce some notation.

Given an $n \times n$ p.f. array A , for $i \in \{1, \dots, n\}$ we define the i -th diagonal of A as follows

$$D_i = \{(i, 1), (i + 1, 2), \dots, (i - 1, n)\}.$$

Here all the arithmetic on the row and column indices is performed modulo n , where $\{1, 2, \dots, n\}$ is the set of reduced residues. The diagonals $D_{i+1}, D_{i+2}, \dots, D_{i+k}$ are called k consecutive diagonals.

Definition 7.6 Let n, k be integers such that $n \geq k \geq 1$. An $n \times n$ p.f. array A is said to be:

- 1) k -diagonal if the non-empty cells of A are exactly those of k diagonals,
- 2) cyclically k -diagonal if the non-empty cells of A are exactly those of k consecutive diagonals.

We recall the following results about solutions of $P(A)$.

Proposition 7.7 ([10]) *Given a cyclically k -diagonal ${}^\lambda\text{NH}_{\mathbb{Z}_v, J}(n; k)$, where J is a subgroup of \mathbb{Z}_v , A , there exists a solution $(\mathcal{R}, \mathcal{C})$ of $P(A)$ in the following cases:*

- 1) if nk is odd, $n \geq k \geq 3$, and $\gcd(n, k - 1) = 1$,
- 2) if nk is odd, $n \geq k$, and $3 < k < 200$,
- 3) if nk is odd, $n \geq k \geq 3$, and $n \geq (k - 2)(k - 1)$.

In all these cases A also admits a pair of compatible orderings.

We note that all the ${}^\lambda\text{NH}_{G, J}(n; k)$ obtained in Theorem 6.23 are cyclically k -diagonal. Therefore, considering these ${}^\lambda\text{NH}_{G, J}(n; k)$'s over cyclic groups \mathbb{Z}_v , we obtain that:

Theorem 7.8 *Let v be an even integer larger than or equal to 42 or an odd one larger than or equal to 29, t be a divisor of v and let $n \geq k \geq 3$ be odd integers such that $v = \frac{2nk}{\lambda} + t$ and that satisfy one of the hypotheses of Proposition 7.7. Then there exists a cellular biembedding of ${}^\lambda K_{(\frac{2nk}{\lambda t} + 1) \times t}$ into an orientable surface whose face lengths are multiples of k strictly larger than k .*

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