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Three coloring via triangle counting

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Abstract

In the first partial result toward Steinberg's now-disproved three coloring conjecture, Abbott and Zhou used a counting argument to show that every planar graph without cycles of lengths 4 through 11 is 3-colorable. Implicit in their proof is a fact about plane graphs: in any plane graph of minimum degree 3, if no two triangles share an edge, then triangles make up strictly fewer than $2/3$ of the faces. We show how this result, combined with Kostochka and Yancey's resolution of Ore's conjecture for $k = 4$, implies that every planar graph without cycles of lengths 4 through 8 is 3-colorable.

In a 1975 letter, Steinberg asked if a planar graph without 4- or 5-cycles is necessarily 3-colorable [10, Problem 9.1]. There was little to no progress on Steinberg's conjecture until 1990. Surely some of this lack of progress was because Steinberg's conjecture is actually false, as established in 2017:

Theorem 1 (Cohen-Addad, Hebdige, Král', Li, and Salgado [6]). *There exists a planar graph without cycles of length* 4 *or* 5 *that is not* 3*-colorable.*

In 1990, Erdős asked [10, Problem 9.2] if there is an integer k such every planar graph without cycles of lengths 4 through k is 3-colorable. The first answer to Erdős's conjecture appeared only a year after he posed it.

Theorem 2 (Abbott and Zhou [1]). *Every planar graph without cycles of lengths* 4 *through* 11 *is* 3*-colorable.*

Abbott and Zhou's proof was at its heart a counting argument. A series of improvements to Theorem 2 have been achieved, all using discharging rather than counting arguments. First, Borodin [3] proved that it suffices to forbid cycles of lengths 4 through 10. Then, Borodin [2] and Sanders and Zhao [9] proved independently that it suffices to forbid cycles of lengths 4 through 9. The current state of the art is the following.

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Theorem 3 (Borodin, Glebov, Raspaud, and Salavatipour [4]). *Every planar graph without cycles of lengths* 4 *through* 7 *is* 3*-colorable.*

Given that Theorem 1 shows that forbidding cycles of lengths 4 and 5 does not ensure a 3-coloring, this leaves an open problem.

Open Problem 4. *If a planar graph does not have cycles of lengths* 4, 5, *or* 6, *is it necessarily* 3*-colorable?*

Our goal in this note is to revisit Abbott and Zhou's proof of Theorem 2 and show how combining their approach with a recent theorem of Kostochka and Yancey yields a result nearly as good as Theorem 3 with very little effort. We begin by making explicit a result about plane graphs that is hidden in Abbott and Zhou's proof of Theorem 2:

Theorem 5. *If G is a connected plane graph of minimum degree* 3 *in which no two triangles share an edge, then triangles make up strictly fewer than* $2/3$ *of its faces.*

Proof. Let G be a connected plane graph with n vertices, e edges, and f faces. Further let n_3 denote the number of degree 3 vertices in G, let f_3 denote the number of triangular faces of G, and let e_3 denote the number of edges that lie on some triangular face. Note that since no two triangles share an edge, $f_3 = e_3/3$. By double counting edges, since the minimum degree of G is 3, we have

$$2e = \sum_{v \in V(G)} \deg v \geq 3n_3 + 4(n - n_3) = 4n - n_3,$$

so $n_3 \geq 4n - 2e$.

Now let v be a vertex of degree 3 in G. Since no edge is contained in two triangles, at least one of the edges incident to v must not be part of a triangle, and so contributes to $e - e_3$. As this edge might be incident to two vertices of degree 3, the most we can claim is that $e - e_3 \geq n_3/2$, or after rearranging, $e_3 \leq e - n_3/2$. Combining this with our inequality on n_3, we have

$$f_3 = \frac{e_3}{3} \leq \frac{e - n_3/2}{3} \leq \frac{2e - 2n}{3} = \frac{2f - 4}{3},$$

where the final equality follows by Euler's formula, $f + n = e + 2$. This proves the result. □

Theorem 5 quickly leads to a proof of Theorem 2:

Proof of Theorem 2. Let G be a plane graph with n vertices, e edges, and f faces, and without cycles of lengths 4 through 11. We prove the result by induction on n, the base case $n = 0$ holding trivially. If G has a vertex v of degree at most 2, then $G - v$ is 3-colorable by induction, and we may extend such a coloring to 3-color G. Thus we may assume that the minimum degree of G is 3. Similarly, we may assume that G is connected.

Let f_3 denote the number of triangles in G. No two triangles of G may share an edge because G does not contain any 4-cycles, so $f_3 < 2f/3$ by Theorem 5. As every edge lies on two faces and every non-triangular face of G has at least 12 edges, the number of non-triangular faces of G satisfies $f - f_3 \leq (2e - 3f_3)/12$. Thus we have

$$f \leq f_3 + \frac{2e - 3f_3}{12} = \frac{e}{6} + \frac{3f_3}{4} < \frac{e}{6} + \frac{f}{2}, \tag{1}$$

so $f < e/3$. By Euler's formula we have $e = n + f - 2$, so

$$e = n + f - 2 < n + \frac{e}{3} - 2, \tag{2}$$

and thus $e < 3n/2 - 3$. This proves that G has average degree less than 3, but that contradicts our assumption that the minimum degree of G is 3, finishing the proof. □

If cycles of length 11 are allowed, then the inequality in (1) must be changed to

$$f \leq f_3 + \frac{2e - 3f_3}{11} = \frac{2e}{11} + \frac{8f_3}{11} < \frac{2e}{11} + \frac{16f}{33}.$$

This implies that $f < 6e/17$, so (2) becomes

$$e = n + f - 2 < n + \frac{6e}{17} - 2,$$

and thus, $e < 17n/11 - 34/11$. This is not enough to guarantee a vertex of degree at most 2, and so the argument used by Abbott and Zhou cannot be used to prove a result stronger than Theorem 2.

There is, however, a different way to use Theorem 5 to prove a result about 3-coloring planar graphs without certain cycles. A graph is *k-critical* if it has chromatic number k, but all of its induced subgraphs have chromatic number strictly less than k. Kostochka and Yancey [8] recently nearly resolved Ore's conjecture on the minimum number of edges in a k-critical graph. They also gave [7] a short and self-contained proof in the case $k = 4$, where the result reduces to the following.

Theorem 6 (Kostochka and Yancey [7, 8])**.** *If G is a 4-critical graph with n vertices and e edges, then*

$$e \geq \frac{5n - 2}{3}.$$

Kostochka and Yancey [7] showed how Theorem 6 leads to a very short proof of Grötsch's celebrated three color theorem (every triangle-free planar graph is 3-colorable). Borodin, Kostochka, Lidický, and Yancey [5] later showed how Theorem 6 can also be used to give a short proof of Grünbaum's three color theorem (every planar graph with at most three triangles is 3-colorable). Below, we use Theorem 6 together with the bound on triangles given by Theorem 5 to derive a result nearly as good as Theorem 3.

Theorem 7. *Every planar graph without cycles of lengths* 4 *through* 8 *is* 3*-colorable.*

Proof. Suppose that the result is not true and take G to be a plane graph of minimal order, say n, that is not 3-colorable despite having no cycles of lengths 4 through 8. Let e denote the number of edges of G and f denote the number of faces. As it is a minimal counterexample, G must be 4-critical, so we have $e \geq 5n/3 - 2/3$ by Theorem 6. Let f_3 denote the number of triangles in G; again we have $f_3 < 2f/3$ by Theorem 5. As the shortest non-triangular faces of G have length 9, the inequality (1) in our proof of Theorem 2 becomes

$$f \leq f_3 + \frac{2e - 3f_3}{9} = \frac{2e}{9} + \frac{2f_3}{3} < \frac{2e}{9} + \frac{4f}{9}.$$

This implies that $f < 2e/5$, so by applying Euler's formula, the inequality (2) becomes

$$e = n + f - 2 < n + \frac{2e}{5} - 2.$$

However, this shows that $e < 5n/3 - 10/3$, which contradicts the fact that $e \geq 5n/3 - 2/3$. ◻

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