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Almost resolvable duplicated Steiner triple systems

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Abstract

We show that there exists a Steiner triple system of order v having a set of almost parallel classes in which each triple occurs exactly twice if and only if $v \equiv 1 \pmod 6$ and $v \notin \{7, 13\}$.

1 Introduction

A *Steiner triple system* of order v, denoted STS(v), is a pair $(V, \mathscr{B})$ where V is a v-set of *points* and $\mathscr{B}$ is a collection of 3-element subsets of V, called *triples*, such that each (unordered) pair of distinct points occurs in exactly one triple. Kirkman [4] proved in 1847 that an STS(v) exists if and only if $v \equiv 1, 3 \pmod 6$. A *partial Steiner triple system* of order v is a pair $(V, \mathscr{B})$ where V is a v-set of points and $\mathscr{B}$ is a collection of triples from V such that each (unordered) pair of distinct points occurs in *at most* one triple.

A *partial parallel class* (PPC) is a set of pairwise disjoint triples of a given point set. Partial parallel classes usually arise in the context of an existing collection $\mathscr{T}$ of triples, and in such cases each triple of the PPC is required to be a triple from $\mathscr{T}$. However, sometimes it will be convenient for us to discuss PPCs solely in terms of a point set, without reference to any specified collection of triples. A PPC that partitions the point set is called a *parallel class*. Given a collection $\mathscr{T}$ of triples, we will call a collection of PPCs which partitions $\mathscr{T}$ a *resolution*. In 1850, Kirkman posed [5] and then solved [6] a problem asking for a resolution of the triples of an STS(15) into parallel classes. An STS(v) with a resolution into parallel classes is now known as a *Kirkman triple system of order* v, denoted KTS(v). Over 100 years after Kirkman found a KTS(15), Lu [8] and Ray-Chaudhuri and Wilson [10] independently proved that a KTS(v) exists if and only if $v \equiv 3 \pmod 6$.

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An STS(v) with $v \equiv 1 \pmod 6$ does not have a parallel class, but one may instead ask for a PPC which covers all but one point. Such a PPC is called an *almost parallel class* (APC) and we call the point it misses the *missed point* of the APC. If one can resolve a set $\mathscr{T}$ of triples into APCs, $\mathscr{T}$ is said to be *almost resolvable*. It can be observed, as in [3], that no STS(v) with $v \equiv 1 \pmod 6$ is almost resolvable (because the number of triples in such an STS(v) is not divisible by the number of triples in an APC).

A *twofold triple system* of order v is a pair $(V, \mathscr{B})$ where V is a v-set of points and $\mathscr{B}$ is a collection of triples from V such that each (unordered) pair of points occurs in exactly two triples of $\mathscr{B}$. In 1974, Hanani [3] proved that there exists an almost resolvable twofold triple system of order v if and only if $v \equiv 1 \pmod 3$; such a system will be denoted ARTTS(v). Observe that in an ARTTS(v) there are v APCs and each point occurs in $v-1$ triples, so it follows that any given point is missed by exactly one APC. In [12], Vanstone et al. prove that there exists an STS(v) which can be resolved into $(v-1)/2$ APCs and one *short* PPC with $(v-1)/6$ triples if and only if $v \equiv 1 \pmod 6$ and $v \notin \{7, 13\}$. An STS(v) with such a resolution is called a *Hanani triple system of order* v, denoted HATS(v). We now define another type of resolvability.

Definition 1.1 Let $(V, \mathscr{B})$ be an STS(v). If the multiset $2\mathscr{B}$ (which contains two copies of each element of $\mathscr{B}$) is almost resolvable, we say that $(V, \mathscr{B})$ is *almost resolvable when duplicated*, and call $(V, \mathscr{B})$ an *almost resolvable duplicated Steiner triple system* of order v, or ARDSTS(v). A resolution of $2\mathscr{B}$ into APCs is called an *ARDSTS resolution* of $(V, \mathscr{B})$.

Example 1.2 The STS(19) with triples given by the orbits of $\{0, 1, 4\}, \{0, 2, 12\}$, and $\{0, 5, 13\}$ under $\mathbb{Z}_{19}$ is an ARDSTS(19). Each of the 19 rows in Table 1 gives an APC and these APCs together form an ARDSTS resolution.

Clearly, if $(V, \mathscr{B})$ is an ARDSTS(v) then $(V, 2\mathscr{B})$ is an ARTTS(v). As with an ARTTS, in an ARDSTS each point is associated with a unique APC which misses that point. The existence of an ARTTS(v) or HATS(v) does not guarantee the existence of an ARDSTS(v). An ARTTS(v) can be obtained from a HATS(v) by constructing a second HATS(v) where the points are relabelled such that the short PPCs of the two HATSs cover a disjoint set of points. The union of these two HATS is an ARTTS (see [12, 2]). If such a relabelling preserves the triples of the original HATS, then the ARTTS resolution is also an ARDSTS resolution. However, no such HATS with an appropriate relabelling has yet been found by the authors. In this paper we construct ARDSTSs of small order, and then build larger systems using Kirkman frames to prove that an ARDSTS(v) exists if and only if $v \equiv 1 \pmod 6$ and $v \notin \{7, 13\}$; see Theorem 2.9.

$\mathcal{P}_0$	4	5	8	16	18	9	12	17	6	10	11	14	1	3	13	2	7	15
$\mathcal{P}_1$	5	6	9	17	0	10	13	18	7	11	12	15	2	4	14	3	8	16
$\mathcal{P}_2$	6	7	10	18	1	11	14	0	8	12	13	16	3	5	15	4	9	17
$\mathcal{P}_3$	7	8	11	0	2	12	15	1	9	13	14	17	4	6	16	5	10	18
$\mathcal{P}_4$	8	9	12	1	3	13	16	2	10	14	15	18	5	7	17	6	11	0
$\mathcal{P}_5$	9	10	13	2	4	14	17	3	11	15	16	0	6	8	18	7	12	1
$\mathcal{P}_6$	10	11	14	3	5	15	18	4	12	16	17	1	7	9	0	8	13	2
$\mathcal{P}_7$	11	12	15	4	6	16	0	5	13	17	18	2	8	10	1	9	14	3
$\mathcal{P}_8$	12	13	16	5	7	17	1	6	14	18	0	3	9	11	2	10	15	4
$\mathcal{P}_9$	13	14	17	6	8	18	2	7	15	0	1	4	10	12	3	11	16	5
$\mathcal{P}_{10}$	14	15	18	7	9	0	3	8	16	1	2	5	11	13	4	12	17	6
$\mathcal{P}_{11}$	15	16	0	8	10	1	4	9	17	2	3	6	12	14	5	13	18	7
$\mathcal{P}_{12}$	16	17	1	9	11	2	5	10	18	3	4	7	13	15	6	14	0	8
$\mathcal{P}_{13}$	17	18	2	10	12	3	6	11	0	4	5	8	14	16	7	15	1	9
$\mathcal{P}_{14}$	18	0	3	11	13	4	7	12	1	5	6	9	15	17	8	16	2	10
$\mathcal{P}_{15}$	0	1	4	12	14	5	8	13	2	6	7	10	16	18	9	17	3	11
$\mathcal{P}_{16}$	1	2	5	13	15	6	9	14	3	7	8	11	17	0	10	18	4	12
$\mathcal{P}_{17}$	2	3	6	14	16	7	10	15	4	8	9	12	18	1	11	0	5	13
$\mathcal{P}_{18}$	3	4	7	15	17	8	11	16	5	9	10	13	0	2	12	1	6	14

Table 1: An ARDSTS resolution of an STS(19).

2 ARDSTS constructions

Motivated by Example 1.2, observe that we can construct an ARDSTS(v) by finding a particular APC in a cyclic STS(v).

Definition 2.1 Let $v \equiv 1 \pmod 6$ and let $\mathcal{P}$ be an APC of $\mathbb{Z}_v$ such that:

1. the union of the orbits under $\mathbb{Z}_v$ of the triples of $\mathcal{P}$ forms an STS(v), and
2. the orbit under $\mathbb{Z}_v$ of each triple of $\mathcal{P}$ contains exactly two triples from $\mathcal{P}$.

Then we call $\mathcal{P}$ a *starter APC*.

Proposition 2.2 *If $\mathcal{P}$ is a starter APC on $\mathbb{Z}_v$, then the orbit of $\mathcal{P}$ under $\mathbb{Z}_v$ yields an ARDSTS(v).*

Proof: Suppose $\mathcal{P}$ is a starter APC on $\mathbb{Z}_v$. Let $(V, \mathscr{B})$ be the STS(v) formed by the union of the orbits under $\mathbb{Z}_v$ of the triples of $\mathcal{P}$. The orbit of $\mathcal{P}$ under $\mathbb{Z}_v$ contains v APCs, and condition (2) in Definition 2.1 ensures that each triple in $\mathscr{B}$ occurs in exactly two of these APCs. $\square$

2.1 Small order existence

We now present some existence results for ARDSTSs of small order. The unique STS(7) has no APC, and so there is no ARDSTS(7). We now prove there is no ARDSTS(13). Up to isomorphism, there are two STS(13)s; one of these is cyclic and the other is not [9]. Before proving that neither of them is an ARDSTS, we prove the following lemma.

Lemma 2.3 (cf. Lemma 3.2 in [1]) *Let $\mathcal{P}$ and $\mathcal{Q}$ be a pair of distinct APCs in a partial Steiner triple system of order v. Then there are at least three triples in $\mathcal{P} \setminus \mathcal{Q}$.*

Proof: Let $\mathcal{R}$ be the symmetric difference of $\mathcal{P}$ and $\mathcal{Q}$. Then $\mathcal{R}$ forms two APCs in a partial Steiner triple system of order $3\,|\mathcal{P} \setminus \mathcal{Q}| + 1$. It is easy to see that any partial Steiner triple system of order 4 or 7 has at most one APC. Thus, $|\mathcal{P} \setminus \mathcal{Q}|$ is at least three. □

Lemma 2.4 *The cyclic STS(13) is not an ARDSTS.*

Proof: Consider the cyclic STS(13) with triples given by the orbits of $\{0, 1, 4\}$ and $\{0, 2, 7\}$ under $\mathbb{Z}_{13}$. We begin by showing that the only APCs in this STS are those in the orbit of

$$\{\{2, 4, 9\}, \{3, 5, 10\}, \{12, 1, 6\}, \{7, 8, 11\}\}$$

under $\mathbb{Z}_{13}$. If $\mathcal{P}$ is any APC in the cyclic STS(13), then the orbit of $\mathcal{P}$ under $\mathbb{Z}_{13}$ is a set of 13 distinct APCs. Thus, if the cyclic STS(13) has more than 13 APCs, then it has at least 26 APCs, and it follows from this that there is a triple B of the STS that occurs in at least 4 distinct APCs. By Lemma 2.3, this implies that there are at least 12 distinct triples, other than B itself, occurring in APCs with B. However, simple counting shows that in any STS(13) there are exactly 10 triples that are disjoint from any given triple. Hence, the cyclic STS(13) has exactly 13 APCs, and as these do not form an ARDSTS resolution (triples in the orbit of $\{0, 1, 4\}$ occur only once and triples in the orbit of $\{0, 2, 7\}$ occur three times), it follows that the cyclic STS(13) is not an ARDSTS. □

Lemma 2.5 *The non-cyclic STS(13) is not an ARDSTS.*

Proof: The triples below form the non-cyclic STS(13), as it is presented in [9].

1	2	3	1	4	5	1	6	7	1	8	9	1	10	11
1	12	13	2	4	6	2	5	7	2	8	10	2	9	12
2	11	13	3	4	8	3	5	12	3	6	13	3	7	11
3	9	10	4	7	9	4	10	13	4	11	12	5	6	10
5	8	11	5	9	13	6	8	12	6	9	11	7	8	13
7	10	12												

Suppose for contradiction that there is an ARDSTS resolution of the non-cyclic STS(13), and let $\mathcal{P}_1$ be the APC missing 1. Since $\{1, 2, 3\}$ is a triple, the points 2 and 3 are covered by distinct triples in $\mathcal{P}_1$. In Table 2 we list the 15 possible pairs of triples that cover 2 and 3.

It can be checked that in each case the required two further triples to cover the remaining six points do not exist. Thus, there is no APC missing 1, and the non-cyclic STS(13) is not an ARDSTS(v).

triple on 2			triple on 3			triples on remaining points					
2	4	6	3	5	12	7	8	13			
2	4	6	3	7	11	5	9	13			
2	4	6	3	9	10	5	8	11	7	8	13
2	5	7	3	4	8	6	9	11			
2	5	7	3	6	13	4	11	12			
2	5	7	3	9	10	4	11	12	6	8	12
2	8	10	3	5	12	4	7	9	6	9	11
2	8	10	3	6	13	4	7	9	4	11	12
2	8	10	3	7	11	5	9	13			
2	9	12	3	4	8	5	6	10			
2	9	12	3	6	13	5	8	11			
2	9	12	3	7	11	4	10	13	5	6	10
2	11	13	3	4	8	5	6	10	7	10	12
2	11	13	3	5	12	4	7	9			
2	11	13	3	9	10	6	8	12			

Table 2: All possible pairs of triples that cover 2 and 3.

□

Lemma 2.6 *There does not exist an ARDSTS(v) for $v = 7$ or 13, but there exists an ARDSTS(v) for all $v \equiv 1 \pmod 6$ where $19 \leq v \leq 85$ or $v = 103$.*

Proof: We have noted previously that there is no ARDSTS(7), and the non-existence of an ARDSTS(13) follows from Lemma 2.4 and Lemma 2.5. Example 1.2 demonstrates existence for $v = 19$, and for each $v \in \{25, 31, 37, 43, 49, 55, 61, 67, 73, 79, 85, 103\}$, a starter APC on $\mathbb{Z}_v$ is given below. Pairs of triples from the same orbit under $\mathbb{Z}_v$ are listed in the same row, together with the elements of $\mathbb{Z}_v$ that map one element of the pair to the other.

ARDSTS(25)

0	1	4	12	13	16	(±12)
8	10	18	22	24	7	(±11)
6	11	20	14	19	3	(±8)
17	23	5	21	2	9	(±4)

ARDSTS(31)

0	1	14	16	17	30	(±15)
2	4	10	21	23	29	(±12)
24	27	5	25	28	6	(±1)
9	13	20	15	19	26	(±6)
3	8	18	7	12	22	(±4)

ARDSTS(37)

0	1	17	2	3	19	(±2)
4	6	14	16	18	26	(±12)
12	15	27	33	36	11	(±16)
20	24	31	21	25	32	(±1)
8	13	22	30	35	7	(±15)
23	29	5	28	34	10	(±5)

ARDSTS(43)

0	1	10	2	3	12	(±2)
4	6	23	5	7	24	(±1)
17	20	31	25	28	39	(±8)
15	19	35	18	22	38	(±3)
16	21	29	27	32	40	(±11)
8	14	26	36	42	11	(±15)
30	37	9	34	41	13	(±4)

ARDSTS(49)

0	1	11	7	8	18	(±7)
3	5	24	4	6	25	(±1)
17	20	32	28	31	43	(±11)
12	16	34	41	45	14	(±20)
30	35	44	33	38	47	(±3)
23	29	46	36	42	10	(±13)
2	9	22	19	26	39	(±17)
13	21	37	40	48	15	(±22)

ARDSTS(55)

0	1	13	2	3	15	(±2)
4	6	28	5	7	29	(±1)
8	11	26	18	21	36	(±10)
16	20	41	23	27	48	(±7)
17	22	33	34	39	50	(±17)
19	25	45	43	49	14	(±24)
30	37	47	35	42	52	(±5)
24	32	51	38	46	10	(±14)
31	40	54	44	53	12	(±13)

ARDSTS(61)

0	1	28	7	8	35	(±7)
9	11	25	10	12	26	(±1)
53	56	18	55	58	20	(±2)
19	23	36	47	51	3	(±28)
16	21	41	24	29	49	(±8)
34	40	52	39	45	57	(±5)
31	38	60	37	44	5	(±6)
14	22	33	46	54	4	(±29)
6	15	30	50	59	13	(±17)
17	27	48	32	42	2	(±15)

ARDSTS(67)

0	1	15	16	17	31	(±16)
25	27	54	59	61	21	(±33)
4	7	23	36	39	55	(±32)
10	14	40	28	32	58	(±18)
30	35	52	19	24	41	(±11)
47	53	11	3	9	34	(±23)
43	50	63	42	49	62	(±1)
12	20	44	37	45	2	(±25)
64	6	18	51	60	5	(±13)
65	8	26	38	48	66	(±27)
22	33	56	46	57	13	(±24)

ARDSTS(73)

0	1	16	9	10	25	(±9)
57	59	15	53	55	11	(±4)
38	41	58	2	5	22	(±36)
19	23	51	24	28	56	(±5)
3	8	26	47	52	70	(±29)
31	37	64	12	18	45	(±19)
27	34	48	65	72	13	(±35)
60	68	21	35	43	69	(±25)
40	49	62	20	29	42	(±20)
36	46	71	4	14	39	(±32)
50	61	7	33	44	63	(±17)
67	6	30	54	66	17	(±13)

ARDSTS(79)

0	1	18	12	13	30	(±12)
5	7	39	6	8	40	(±1)
49	52	71	59	62	2	(±10)
11	15	46	64	68	20	(±26)
31	36	57	77	3	24	(±33)
70	76	27	72	78	29	(±2)
38	45	61	44	51	67	(±6)
17	25	54	58	66	16	(±38)
19	28	43	41	50	65	(±22)
37	47	75	73	4	32	(±36)
23	34	48	63	74	9	(±39)
14	26	53	21	33	60	(±7)
22	35	55	56	69	10	(±34)

ARDSTS(85)

0	1	39	51	52	5	(±34)
81	83	18	82	84	19	(±1)
32	35	68	34	37	70	(±2)
23	27	46	65	69	3	(±42)
58	63	10	74	79	26	(±16)
25	31	49	53	59	77	(±28)
9	16	44	29	36	64	(±20)
20	28	45	47	55	72	(±27)
21	30	61	33	42	73	(±12)
14	24	40	50	60	76	(±36)
2	13	43	56	67	12	(±31)
66	78	8	80	7	22	(±14)
4	17	38	41	54	75	(±37)
48	62	6	57	71	15	(±9)

ARDSTS(103)

0	55	45	72	24	14	(±31)
21	57	13	41	77	33	(±20)
101	74	4	102	75	5	(±1)
23	69	70	68	11	12	(±45)
81	98	20	89	3	28	(±8)
46	59	66	99	9	16	(±50)
64	80	49	76	92	61	(±12)
83	71	17	100	88	34	(±17)
39	48	37	84	93	82	(±45)
25	44	78	43	62	96	(±18)
50	10	36	87	47	73	(±37)
35	65	97	95	22	54	(±43)
2	31	7	27	56	32	(±25)
38	42	60	63	67	85	(±25)
18	15	53	94	91	26	(±27)
1	29	52	58	86	6	(±46)
40	19	79	51	30	90	(±11)

□

2.2 Large orders

To construct ARDSTSs of larger orders we use Kirkman frames (see [2]). A *Kirkman frame* consists of a set V of points that is partitioned into *groups*, a set of triples of points such that each pair of points from distinct groups occurs in exactly one triple and each pair of points from the same group occurs in zero triples, and a resolution of the triples into PPCs such that each PPC partitions $V \setminus G$ for some group G. A Kirkman frame is said to be of *type* $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ if there are u_i groups of cardinality g_i for $i = 1, 2, \dots, s$ (and no other groups). In a Kirkman frame with point set V, a PPC that partitions $V \setminus G$ is called a PPC with *hole* G. It is known that in any Kirkman frame, the number of PPCs having hole G is $|G|/2$, see [11]. We use this last result in the proof of Lemma 2.7.

Lemma 2.7 *If there exists a Kirkman frame of type* $g_1^{u_1} g_2^{u_2} \dots g_s^{u_s}$ *and an ARDSTS*$(g_i + 1)$ *for* $i = 1, 2, \dots, s$*, then there exists an ARDSTS*(v) *with* $v = u_1 g_1 + u_2 g_2 + \dots + u_w g_w + 1$.

Proof: The points of the ARDSTS(v) are the points of the Kirkman frame together with a new point ∞. For each group G of the Kirkman frame, we place a copy of an ARDSTS$(|G| + 1)$ on $G \cup \{\infty\}$. The triples of these ARDSTSs together with the triples of the Kirkman frame form an STS(v). We show that this is an ARDSTS(v).

For each group G, the ARDSTS on $G \cup \{\infty\}$ has an APC missing ∞, and the union (over the groups) of these is an APC missing ∞ in our ARDSTS(v). For each point x of the Kirkman frame, if G is the group containing x, then we take the APC missing x of the ARDSTS on $G \cup \{\infty\}$ together with a PPC with hole G of the Kirkman frame. This yields an APC missing x in our ARDSTS(v). Since there are exactly $|G|/2$ PPCs with hole G, we can pair these PPCs with the points of G such that each PPC with hole G is used exactly twice. It can be seen that the resulting APCs form an ARDSTS resolution. □

The existence problem for Kirkman frames of type t^u was settled by Stinson [11], and for Kirkman frames of type $h^u m^1$ with $h \equiv 0 \pmod{12}$ by Wei and Ge [13]. We only need such frames as given by the following result.

Theorem 2.8 ([11, 13]) *For all $u \geq 4$, there exist Kirkman frames of each of the following types.*

$$18^u \qquad 24^u \qquad 24^u 18 \qquad 24^u 30 \qquad 24^u 36$$

Theorem 2.9 *There exists an ARDSTS(v) if and only if $v \equiv 1 \pmod 6$ and $v \notin \{7, 13\}$.*

Proof: The non-existence of an ARDSTS(7) and an ARDSTS(13) is given by Lemma 2.6. We need to show that there exists an ARDSTS(v) for all $v \equiv 1 \pmod 6$ with $v \geq 19$. The proof splits into four cases depending on whether $v \equiv 1, 7, 13$ or $19 \pmod{24}$. In each case we obtain the required ARDSTSs by applying Lemma 2.7 using Kirkman frames given by Theorem 2.8 and ARDSTSs given by Lemma 2.6.

1. Suppose $v \equiv 1 \pmod{24}$. Let $v = 24u + 1$ with $u \geq 1$. For $u \leq 3$ we have $v \in \{25, 49, 73\}$ and an ARDSTS(v) is given by Lemma 2.6. For $u \geq 4$ we use a Kirkman frame of type 24^u and an ARDSTS(25) in Lemma 2.7.
2. Suppose $v \equiv 7 \pmod{24}$. Let $v = 24u + 31$ with $u \geq 0$. For $u \leq 3$ we have $v \in \{31, 55, 79, 103\}$ and an ARDSTS(v) is given by Lemma 2.6. For $u \geq 4$ we use a Kirkman frame of type $24^u 30^1$, an ARDSTS(25), and an ARDSTS(31) in Lemma 2.7.
3. Suppose $v \equiv 13 \pmod{24}$. Let $v = 24u + 37$ with $u \geq 0$. For $u \leq 2$ we have $v \in \{37, 61, 85\}$ and an ARDSTS(v) is given by Lemma 2.6. For $u = 3$ we use a Kirkman frame of type 18^6 and an ARDSTS(19) in Lemma 2.7. For $u \geq 4$ we use a Kirkman frame of type $24^u 36^1$, an ARDSTS(25), and an ARDSTS(37) in Lemma 2.7.
4. Suppose $v \equiv 19 \pmod{24}$. Let $v = 24u + 19$ with $u \geq 0$. For $u \leq 2$ we have $v \in \{19, 43, 67\}$ and an ARDSTS(v) is given by Lemma 2.6. For $u = 3$ we use a Kirkman frame of type 18^5 and an ARDSTS(19) in Lemma 2.7. For $u \geq 4$ we use a Kirkman frame of type $24^u 18^1$, an ARDSTS(25), and an ARDSTS(19) in Lemma 2.7.

□

3 Further considerations

We mention some open problems that arise from the work in this paper. We have seen that there exists a starter APC on $\mathbb{Z}_v$ for all $v \equiv 1 \pmod 6$ in the range $19 \leq v \leq 85$ and for $v = 103$. So it is natural to ask the following question.

(1) Does there exist a starter APC on $\mathbb{Z}_v$ for all $v \equiv 1 \pmod 6$ with $v \geq 19$?

Two PPCs that intersect in at most one triple are said to be *orthogonal*, and there are many results on triple systems that consider orthogonality of PPCs, see [2, 7]. One may ask about the existence of ARDSTSs in which every pair of APCs is orthogonal. We call such an ARDSTS *self-orthogonal*.

It is straightforward to determine whether an ARDSTS(v) constructed from a starter APC is self-orthogonal. A starter APC consists of $r = (v-1)/6$ pairs of triples where the two triples from each pair are from the same orbit under $\mathbb{Z}_v$. If we let these pairs of triples be $\{T_1, T_1'\}, \{T_2, T_2'\}, \ldots, \{T_r, T_r'\}$, and let $T_i + x_i = T_i'$ for $i = 1, 2, \ldots, r$, then it is easy to see that the constructed ARDSTS(v) is self-orthogonal if and only if $\pm x_1, \pm x_2, \ldots, \pm x_r$ are pairwise distinct. For each starter APC in the proof of Lemma 2.6, the two triples in each row are from the same orbit, and the corresponding values $\pm x_1, \pm x_2, \ldots, \pm x_r$ are given in the right-most column.

Thus, it can be seen that the ARDSTS(v)s with $19 \leq v \leq 85$ given in Example 1.2 and the proof of Lemma 2.6 are all self-orthogonal, but the ARDSTS(103) in the proof of Lemma 2.6 is not. Furthermore, it is easy to see that an ARDSTS constructed using Kirkman frames, as in Lemma 2.7, is not (in general) self-orthogonal because if G is any hole of the frame, then the constructed ARDSTS has two APCs whose intersection is a PPC of the frame with hole G. The following two questions arise.

(2) Does there exist a self-orthogonal ARDSTS(v) for all $v \equiv 1 \pmod 6$ with $v \geq 19$?

(3) Does there exist a starter APC for a self-orthogonal ARDSTS(v) for all $v \equiv 1 \pmod 6$ with $v \geq 19$?

Finally, there is the question (alluded to earlier) regarding the existence of a HATS(v) having an automorphism θ such that the union of the short PPC and its image under θ is an APC, which would yield an ARDSTS(v).

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