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Bounds on the outer-independent Roman domination number of unicyclic and bicyclic graphs

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Abstract

A function $f : V(G) \longrightarrow \{0, 1, 2\}$ is called a Roman dominating function on a graph G if for every vertex $v \in V(G)$ with $f(v) = 0$, there exists a vertex u adjacent to v with $f(u) = 2$. A Roman dominating function f is an outer-independent Roman dominating function if the set $\{v \in V(G) : f(v) = 0\}$ is independent. The outer-independent Roman domination number of G is the minimum weight of an outer-independent Roman dominating function on G. Chellali and Dehgardi [*Commun. Comb. Optim.* 6 (2021), 273–286] proved that the outer-independent Roman domination number of any tree T of order $n \geq 3$ is bounded above by $5n/6$. In this paper, aiming to obtain best upper bounds for the outer-independent Roman domination number in cactus graphs, we prove an $8n/9$-upper bound for unicyclic graphs and a $9n/10$-upper bound for bicyclic graphs. We also characterize extremal unicyclic graphs as well as bicyclic graphs, achieving equality for the given bounds.

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1 Introduction

For graph theory notation and terminology not given here we refer to [9]. We consider finite and simple graphs G with vertex set $V = V(G)$ and edge set $E(G)$. The number of vertices of G is called the *order* of G and is denoted by $n = n(G)$. The *open neighborhood* of a vertex $v \in V$ is $N(v) = N_G(v) = \{u \in V \mid uv \in E\}$ and the *closed neighborhood* of v is $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex v, denoted by $\deg(v)$ (or $\deg_G(v)$ to refer to G), is the cardinality of its open neighborhood. We denote by $\delta(G)$ and $\Delta(G)$, the minimum and maximum degrees among all vertices of G, respectively. A vertex of degree one is referred as a *leaf* and a vertex adjacent to a leaf is referred as a *support* vertex. A *strong support vertex* is a support vertex adjacent to at least two leaves, and a *weak support vertex* is a support vertex adjacent to exactly one leaf. A *component* of a graph G is a maximal connected subgraph of G. We denote by P_n, C_n and $K_{m,n}$ the path of order n, the cycle of order n, and the complete bipartite graph such that one partite set has m vertices and the other partite set has n vertices, respectively. We refer to $K_{1,3}$, as a *claw*. For a subset S of vertices of a graph G, we denote by $G[S]$ the subgraph of G induced by S. A graph G is *claw-free* if $G[S] \ncong K_{1,3}$ for any set S of cardinality 4. A *unicyclic graph* is a graph obtained from a tree by adding precisely one edge. Equivalently, a unicyclic graph is a graph with precisely one cycle. A *bicyclic graph* is a graph with precisely two cycles. A *cactus graph* is a graph such that no pair of distinct cycles have a common edge. An *independent set* in a graph G is a subset S of vertices such that the subgraph induced by S has no edges.

A function $f : V \longrightarrow \{0, 1, 2\}$ having the property that for every vertex $v \in V$ with $f(v) = 0$, there exists a vertex $u \in N(v)$ with $f(u) = 2$, is called a *Roman dominating function* or just an RDF. The *weight* of an RDF f is the sum $f(V) = \sum_{v \in V} f(v)$. The minimum weight of an RDF on G is called the *Roman domination number* of G and is denoted by $\gamma_R(G)$. The mathematical concept of Roman domination was developed by Cockayne et al. [8]. Many variations, generalizations and applications of Roman domination parameters have been studied, and to see the latest progress until 2020 see [5, 6, 7].

Ahangar et al. [1] introduced the concept of outer-independent Roman domination in graphs. An RDF f in a graph G is an *outer-independent Roman dominating function* (OIRDF) on G if the set $\{v \in V(G) : f(v) = 0\}$ is an independent set. The *outer-independent Roman domination number* $\gamma_{oiR}(G)$ is the minimum weight of an OIRDF on G. The concept of outer-independent Roman domination in graphs was further studied in, for example, [2, 3, 10, 11, 12, 13]. Chellali and Dehgardi [4] proved that $\gamma_{oiR}(T) \leq 5n/6$ in any tree T of order n, and they characterized trees achieving equality for this bound.

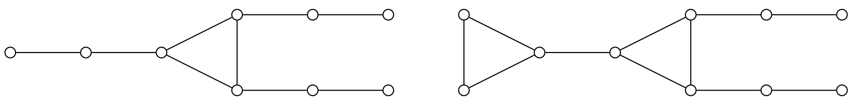
In this paper we present upper bounds for the outer-independent Roman domination number in unicyclic graphs as well as bicyclic graphs, and characterize unicyclic graphs and bicyclic graphs achieving equality for the given bounds. The organization of the paper is as follows. In Section 2, we present our main results, namely Propositions 2.1 and 2.2, and Theorems 2.3 and 2.4. In Section 3, we present a proof for Propositions 2.1 and 2.2. In Section 4, we present a proof for Theorem 2.3. In Section 5, we present a proof for Theorem 2.4. In Section 6, we propose our suggested bounds and conjectures for cactus graphs. We make use of the following.

Theorem 1.1 (Chellali et al. [4]) *For any tree T of order $n \geq 3$, $\gamma_{oiR}(T) \leq 5n/6$.*

For an RDF f in a graph G, we denote by V_i (or V_i^f to refer to f) the set of all vertices of G with label i under f. Thus an RDF f can be represented by a triple (V_0, V_1, V_2), and we can use the notation $f = (V_0, V_1, V_2)$.

2 Main results

Let H_1 and H_2 be the graphs depicted in Figure 1.



H_1 H_2

Figure 1. Graphs H_1 and H_2.

We will prove the following.

Proposition 2.1 *For any integer $n \geq 3$, $\gamma_{oiR}(C_n) = \frac{3n+j}{4}$ if $n \equiv j \pmod 4$, $j = 0, 1, 2, 3$.*

Proposition 2.2 *For any integer $n \geq 3$, $\gamma_{oiR}(P_n) = \frac{3n+j}{4}$ if $n \equiv j \pmod 4$, $j = 0, 1, 2$, and $\gamma_{oiR}(P_n) = \frac{3n-1}{4}$ if $n \equiv 3 \pmod 4$.*

Theorem 2.3 *For a unicyclic graph $G \neq K_3$, $\gamma_{oiR}(G) \leq 8n/9$, with equality if and only if $G = H_1$, where H_1 is depicted in Figure 1.*

Theorem 2.4 *For a bicyclic graph G of order n, $\gamma_{oiR}(G) \leq 9n/10$, with equality if and only if $G = H_2$, where H_2 is depicted in Figure 1.*

3 Proof of Propositions 2.1 and 2.2

We only present the proof of Proposition 2.1; the proof of Proposition 2.2 is similar and is thus omitted.

Let $V(C_n) = \{v_1, \ldots, v_n\}$, where v_i is adjacent to v_{i+1} for $i = 1, 2 \ldots, n-1$ and v_1 is adjacent to v_n. It is evident that $\gamma_{oiR}(C_n) < n$ for $n \geq 4$, since assigning 2 to v_2, 0 to

v_1 and v_3, and 1 to all other vertices yields an OIRDF. It is easy to see that $\gamma_{oiR}(C_3) = 3$, $\gamma_{oiR}(C_4) = 3$, $\gamma_{oiR}(C_5) = 4$, $\gamma_{oiR}(C_6) = 5$ and $\gamma_{oiR}(C_7) = 6$. Thus, assume that $n \geq 8$. We show that for each n, $\gamma_{oiR}(C_n) = \gamma_{oiR}(C_{n-4})+3$, and then the result follows by induction on n. Let $f = (V_0, V_1, V_2)$ be a $\gamma_{oiR}(C_n)$-function such that $|V_2|$ is as small as possible. If $V_2 = \emptyset$, then $w(f) = n$, a contradiction. Thus, $V_2 \neq \emptyset$. Clearly, for any vertex $v_i \in V_2$, $|\{v_{i-1}, v_{i+1}\} \cap V_0| \geq 1$. Suppose that there is a vertex $v_i \in V_2$ such that $|\{v_{i-1}, v_{i+1}\} \cap V_0| = 1$. Assume that $\{v_{i-1}, v_{i+1}\} \cap V_0 = \{v_{i-1}\}$. Then we change both $f(v_i)$ and $f(v_{i-1})$ to 1 to obtain a $\gamma_{oiR}(C_n)$-function with fewer vertices assigned 2 than under f, contradicting the choice of f. Thus, $|\{v_{i-1}, v_{i+1}\} \cap V_0| = 2$, for each vertex $v_i \in V_2$. Let $v_i \in V_2$. As noted, $\{v_{i-1}, v_{i+1}\} \subseteq V_0$. Observe that $v_{i+2} \notin V_0$ and $v_{i-2} \notin V_0$. Suppose that $v_{i+2} \in V_2$. Then we change both $f(v_i)$ and $f(v_{i-1})$ to 1, leading as before to a contradiction on the choice of f. Thus, $v_{i+2} \in V_1$, and likewise, $v_{i-2} \in V_1$. Let G' be obtained from C_n by removing v_{i-1}, v_i, v_{i+1} and v_{i+2}, and then joining v_{i-2} to v_{i+3}. Then $G' = C_{n-4}$ and $f|_{G'}$ is an OIRDF for G', implying that $\gamma_{oiR}(G') = \gamma_{oiR}(C_{n-4}) \leq \gamma_{oiR}(C_n) - 3$, since $f(v_i) + f(v_{i+2}) = 3$.

On the other hand, let $g = (V_0, V_1, V_2)$ be a $\gamma_{oiR}(C_{n-4})$-function such that $|V_2|$ is as small as possible. If $V_1 = \emptyset$, then $n - 4$ is even (since by the choice of g, both neighbors of a vertex of V_0 belong to V_2 and both neighbors of a vertex of V_2 belong to V_0) and we may assume that $V_2 = \{v_{2i+1} : 0 \leq i < \frac{n-4}{2}\}$ and $V_0 = \{v_{2i} : 1 \leq i \leq \frac{n-4}{2}\}$, and so $\gamma_{oiR}(C_{n-4}) = n - 4$, a contradiction, since $n \geq 8$. Thus, there is a vertex v_j such that $v_j \in V_1$. Then we form a graph G'' by replacing v_i with a path $abcde$, and join a to v_{i-1} and e to v_{i+1}. Observe that $G'' = C_n$. Then h defined on G'' by $h(x) = g(x)$ if $x \notin \{a, b, c, d, e\}$, $h(a) = h(e) = 1$, $h(b) = h(d) = 0$ and $h(c) = 2$, is an OIRDF on G'', and so $\gamma_{oiR}(G'') \leq w(g) + 3 = \gamma_{oiR}(C_{n-4}) + 3$. We deduce that $\gamma_{oiR}(C_n) \leq \gamma_{oiR}(C_{n-4}) + 3$. Hence $\gamma_{oiR}(C_n) = \gamma_{oiR}(C_{n-4}) + 3$. Now it is straightforward to prove that $\gamma_{oiR}(C_n) = \frac{3n+j}{4}$ using induction on n.

4 Proof of Theorem 2.3

We prove this by induction on the order n. For the base step of the induction it is easy to see that the result holds if $n \leq 5$. Thus let $n \geq 6$ and assume the result holds for all unicyclic graph G' of order $5 \leq n' < n$, that is, $\gamma_{oiR}(G') \leq \frac{8n'}{9}$, with equality if and only if $G' = H_1$. Now consider the unicyclic graph G of order n. Assume G has no leaves. Then $G = C_n$. By Proposition 2.1, $\gamma_{oiR}(C_n) = \frac{3n+j}{4}$ if $n \equiv j \pmod 4$, $j = 0, 1, 2, 3$. Since $n \geq 6$, we find that $\frac{3n+j}{4} < 8n/9$. We thus assume that G has at least one leaf. Let C be the unique cycle of G, and x_0 be a vertex of C such that $\deg(x_0)$ is as maximum as possible and $x_0 \dots x_d$, where $d \geq 1$, be a path from x_0 to a farthest leaf x_d of G, where x_i is outside C for $i = 1, \dots, d$. Let $|V(C)| = k$, and $V(C) = \{x_0, y_1, y_2, \dots, y_{k-1}\}$, where $k \geq 3$, x_0 is adjacent to y_1 and y_{k-1}, and y_i is adjacent to y_{i+1} for $i = 1, 2, \dots, k-2$.

Lemma 4.1 *If* $\deg(y_i) = 2$ *for some* $i \in \{1, 2, \dots, k-1\}$, *then* $\gamma_{oiR}(G) < 8n/9$.

Proof. Suppose that $\deg(y_i) = 2$ for some $i \in \{1, 2, \dots, k-1\}$, and let $T = G - y_i$.

By Theorem 1.1, $\gamma_{oiR}(T) \leq 5(n-1))/6$. Then we extend any $\gamma_{oiR}(T)$-function to an OIRDF for G by assigning 1 to y_i. Thus, $\gamma_{oiR}(G) \leq 5(n-1))/6 + 1 < 8n/9$, since $n \geq 6$. □

By Lemma 4.1, we have $\deg(y_i) \geq 3$ for $i = 1, 2, \ldots, k-1$.

Lemma 4.2 *If $v \in C$ is a strong support vertex of degree at least four such that all its neighbors but two are leaves, or $v \in V(G) - C$ is a strong support vertex such that all its neighbors but one are leaves, then $\gamma_{oiR}(G) < 8n/9$.*

Proof. Let $v \in C$ be a strong support vertex of degree at least four such that all its neighbors but two are leaves and let v' and v'' be the neighbors of v in C. Let T' be the tree obtained from G by removal of edges $v'v$ and $v''v$, such that T' contains v'. If $|V(T')| = 2$, then $n = 1 + \deg(v)$ and by assigning 2 to v, 1 to v' and 0 to each other vertex we obtain that $\gamma_{oiR}(G) = 3 < 8n/9$, since $\deg(v) \geq 4$. Thus assume that $|V(T')| \geq 3$. By Theorem 1.1, $\gamma_{oiR}(T') \leq 5(n-(\deg(v)-1))/6$. Then we extend any $\gamma_{oiR}(T')$-function to an OIRDF for G by assigning 2 to v and 0 to its leaf neighbors. Thus, $\gamma_{oiR}(G) \leq 5(n-(\deg(v)-1))/6+2 < 8n/9$, since $\deg(v) \geq 4$. The proof for the case $v \in V(G) - C$ is similar and is omitted. □

We proceed with Lemma 4.3.

Lemma 4.3 *If $d = 1$, then $\gamma_{oiR}(G) < 8n/9$.*

Proof. Assume that $d = 1$. Then $n = 2k$, since $\deg(y_i) \geq 3$, for $i = 1, 2, \ldots, k-1$. It is straightforward to see that if $k = 3$ then $\gamma_{oiR}(G) = 5 < 8n/9$, if $k = 4$, then $\gamma_{oiR}(G) = 6 < 8n/9$, if $k = 5$ then $\gamma_{oiR}(G) = 8 < 8n/9$ and if $k = 6$ then $\gamma_{oiR}(G) = 9 < 8n/9$. Thus assume that $k \geq 7$. Let T be the tree obtained by removing the edges $y_{k-1}x_0$ and y_2y_3 such that T contains y_3. By Theorem 1.1, $\gamma_{oiR}(T) \leq 5(n-6)/6$. We assign 2 to x_0, 1 to y_2 and the leaf-neighbors of y_1 and y_2, and 0 to x_1 and y_1 to extend any $\gamma_{oiR}(T)$-function to an OIRDF for G. Then $\gamma_{oiR}(G) \leq 5(n-6)/6 + 5 < 8n/9$. □

We thus assume that $d \geq 2$. By Lemma 4.2, $\deg(x_{d-1}) = 2$, and any neighbor of x_{d-2}, with the exception of x_{d-3} (if $d \geq 3$) and with the exception of its neighbor on C (if $d = 2$), is a leaf or a support vertex of degree two. We proceed with Lemma 4.4.

Lemma 4.4 *If $d \geq 3$, then $\gamma_{oiR}(G) < 8n/9$.*

Proof. Assume that $d \geq 3$. We consider the following two cases.

Case 1. $\deg(x_{d-2}) \geq 3$. Assume that x_{d-2} has a neighbors as support vertices of degree two and b neighbors as leaves. Clearly $a \geq 1$ and $a + b \geq 2$. Let G' be the component of $G - x_{d-2}$ that contains x_{d-3}. If $G' = K_3$, then $d = 3$ and $n = 2a+b+4$. Then we assign 2 to x_1, 0 to each neighbor of x_1, and 1 to any other vertex of G to obtain that $\gamma_{oiR}(G) \leq a + 4 < 8n/9$, since $a + b \geq 2$. Thus, assume that $G' \neq K_3$. By the inductive hypothesis, $\gamma_{oiR}(G') \leq 8n'/9$. Let f' be a $\gamma_{oiR}(T')$-function. Then we extend f' to an OIRDF for G by assigning 2 to x_{d-2}, 0 to its neighbors which are leaves or support vertices of degree two, and 1 to any other vertex. Since $a \geq 1$ and $a + b \geq 2$, we obtain that

$$\begin{aligned}\gamma_{oiR}(G) &\leq \gamma_{oiR}(G') + 2 + a \\ &\leq \frac{8}{9}(n - 2a - b - 1) + 2 + a \\ &= \frac{8n - 7a - 8b + 10}{9} < \frac{8}{9}n.\end{aligned}$$

Case 2. $\deg(x_{d-2}) = 2$. If x_{d-3} has a neighbor w_1 outside C such that there is a path $x_{d-3}w_1w_2w_3$ from x_{d-3} to a leaf w_3, then w_3 plays the role of x_d, and thus, by Lemma 4.2 and Case 1 of the proof of Lemma 4.4, we may assume that $\deg(w_1) = \deg(w_2) = 2$. Thus, any neighbor of x_{d-3} outside C is a leaf, a support vertex of degree two, or a vertex of degree two which is adjacent to a support vertex of degree two.

Assume that $d \geq 4$. Let G' be the component of $G - x_{d-3}x_{d-4}$ that contains x_{d-4}. It is evident that $G' \neq K_3$. By the inductive hypothesis, $\gamma_{oiR}(G') \leq 8n'/9$. Let f' be a $\gamma_{oiR}(G')$-function. If $\deg(x_{d-3}) = 2$, then $n' = n - 4$, and we extend f' to an OIRDF for G by assigning 2 to x_{d-1}, 0 to x_d and x_{d-2} and 1 to x_{d-3} to obtain $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 3 \leq \frac{8}{9}(n - 4) + 3 < 8n/9$. Thus, assume that $\deg(x_{d-3}) \geq 3$. Assume that x_{d-3} is adjacent to c leaves, b support vertices of degree two and a vertices of degree two each of which is adjacent to a support vertices of degree two (all outside C). Then $a \geq 1$ and $a + b + c \geq 2$. Then we extend f' to an OIRDF for G by assigning 2 to x_{d-3} and each support vertex at distance 2 from x_{d-3}, 1 to any leaf at distance two from x_{d-3} and 0 to any other vertex in $V(G) - V(G')$. Then

$$\begin{aligned}\gamma_{oiR}(G) &\leq \gamma_{oiR}(G') + 2 + 2a + b \\ &\leq \frac{8}{9}(n - 3a - 2b - c - 1) + 2 + 2a + b \\ &= \frac{8n - 6a - 7b - 8c + 10}{9} < \frac{8}{9}n,\end{aligned}$$

since $a + b + c \geq 2$.

Next assume that $d = 3$. Let G' be the component of $G - \{x_0y_1, x_0y_{k-1}\}$ that contains y_1. If $G' \neq K_2$, then by Theorem 1.1, $\gamma_{oiR}(G') \leq 8n'/9$, and as before, we can extend a $\gamma_{oiR}(G')$-function to an OIRDF for G to obtain that $\gamma_{oiR}(G) < 8n/9$. Thus, assume that $G' = K_2$. Assume that x_0 is adjacent to c leaves, b support vertices of degree two and a vertices of degree two each of which is adjacent to a support vertex of degree two. Then $n = 3a + 2b + c + 3$, $a \geq 1$ and $a + b + c \geq 1$. Then we define a function f on $V(G)$ by assigning 2 to x_{d-3} and each support vertex at distance 2 from x_{d-3}, 1 to y_1 and to any leaf at distance two from x_{d-3} and 0 to any other vertex in $V(G)$. Then $\gamma_{oiR}(G) \leq 3 + 2a + b < \frac{8}{9}(3a + 2b + c + 3) = \frac{8}{9}n$, as desired. □

By Lemmas 4.3 and 4.4, we may assume for the next that $d = 2$. Furthermore any neighbor of x_0 outside C is a leaf or a support vertex of degree two. Let a be the number of support neighbors of x_0 (outside C) and b be the number of leaf neighbors of x_0 (outside C).

Assume that $\deg(x_0) \geq 4$. Then $a+b \geq 2$. Let G' be the component of $G-x_0$ that contains y_1. Then G' is a tree of order at least four. By Theorem 1.1, $\gamma_{oiR}(G') \leq 5n'/6$. Let f' be a $\gamma_{oiR}(T')$-function. Then we extend f' to an OIRDF for T by assigning 2 to x_0, 0 to its neighbors outside C, and 1 to any other vertex, and thus we can obtain that $\gamma_{oiR}(G) \leq \frac{5}{6}(n-2a-b-1)+a+2 < \frac{8}{9}n$. We thus assume that $\deg(x_0) = 3$. By the choice of x_0, any vertex on C is of degree three which either is a support vertex or is adjacent to a support vertex of degree two.

Assume that y_i is a support vertex for some $i \in \{1, 2, \ldots, k-1\}$ and z_i is the leaf adjacent to y_i. Let $T = G - \{y_i, z_i\}$. Then $|V(T)| \geq 5$. By Theorem 1.1, $\gamma_{oiR}(T) \leq 5(n-2)/6$. Then we extend any $\gamma_{oiR}(T)$-function to an OIRDF for T by assigning 1 to both y_i and z_i, and so $\gamma_{oiR}(G) \leq \frac{5}{6}(n-2)+2 < \frac{8}{9}n$. We thus assume that y_i is a vertex of degree three adjacent to a support vertex of degree two, for $i = 1, 2, \ldots, k-1$.

Assume that $k \geq 5$. Let T be the tree containing y_2 that is obtained by removing the edges y_1y_2 and $y_{k-1}x_0$. Note that $|V(T) \geq 6$. Let G' be the unicyclic graph obtained from T by joining y_2 to y_{k-1}. By the inductive hypothesis, $\gamma_{oiR}(G') \leq \frac{8}{9}n'$. Let f' be a $\gamma_{oiR}(G')$-function. Clearly $f'(y_2) \neq 0$ or $f'(y_{k-1}) \neq 0$, since they are adjacent in G'. Without loss of generality, assume that $f'(y_2) \neq 0$. Let y_1' be the support vertex adjacent to y_2, and let y_1'' be the leaf adjacent to y_1'. Clearly we may assume that $f(y_1') + f(y_1'') = 2$. Let f be a function defined on $V(G)$ by $f(x_0) = 2$, $f(x) = f'(x)$ if $x \in V(G') - \{y_1', y_1''\}$, $f(x) = 0$ if $x \in \{x_1, y_1, y_1''\}$, $f(y_1') = 2$ and $f(x) = 1$ otherwise. Then f is an OIRDF for G, and so $\gamma_{oiR}(G) \leq \frac{8}{9}(n-6)+5 < \frac{8}{9}n$.

If $k = 4$, then $n = 12$. Let f be a function defined on $V(G)$ by $f(x_0) = 2$, $f(x) = 0$ if x is adjacent to x_0, and $f(x) = 1$ otherwise. Then f is an OIRDF for G with weight $2 + (n-4) = n-2$, and clearly, $\gamma_{oiR}(G) \leq n-2 < \frac{8}{9}n$. Thus, we assume that $k = 3$. Then $n = 9$ and $G = H_1$. Let f be a function defined on $V(G)$ by $f(x_1) = 2$, $f(x_0) = f(x_2) = 0$, and $f(x) = 1$ otherwise. Then f is an OIRDF for G with weight $8 = \frac{8}{9}n$.

We thus have proved the upper bound. If the equality holds, then following the above proof, we deduce that $G = H_1$.

5 Proof of Theorem 2.4

We prove this theorem by induction on the order n. Clearly, $n \geq 5$. For the base step of the induction, if $n \leq 9$ then we choose a vertex x with two non-adjacent neighbors y and z, and assign 2 to x, 0 to both y and z, and 1 to other vertices of G, and so $\gamma_{oiR}(G) \leq n-1 < 9n/10$. Thus let $n \geq 10$ and assume that the result holds for all bicyclic graphs of order $n' < n$. Now consider the bicyclic graph G of order n. Let C_1 and C_2 be the cycles of G, and let $V(C_1) = \{v_1, \ldots, v_k\}$ and $V(C_2) = \{u_1, \ldots, u_l\}$. Without loss of generality, assume that $d(C_1, C_2) = d(v_1, u_1)$. Let $d(C_1, C_2) = s$ and P the shortest path between v_1 and u_1. The following lemma can be proved similarly to the proof of Lemma 4.2, and thus we omit its proof.

Lemma 5.1 *If $v \in P \cup C_1 \cup C_2$ is a strong support vertex such that all of its neighbors outside $P \cup C_1 \cup C_2$ are leaves, or $v \in V(G) - P \cup C_1 \cup C_2$ is a strong support vertex such that all of its neighbors but one are leaves, then $\gamma_{oiR}(G) < 9n/10$.*

We proceed with Lemma 5.2.

Lemma 5.2 *If x is a vertex on $P \cup C_1 \cup C_2$ and there is a path of length $d \geq 3$ from x to a furthest leaf x_d from x that intersects $P \cup C_1 \cup C_2$ only in x, then $\gamma_{oiR}(G) < 9n/10$.*

Proof. Assume that there is a path $xx_1x_2 \dots x_d$ to a leaf x_d and $\{x, x_1, \dots, x_d\} \cap (V(P) \cup V(C_1) \cup V(C_2)) = \{x\}$. By Lemma 5.1, $\deg(x_{d-1}) = 2$. Assume that $\deg(x_{d-2}) \geq 3$. Then we may assume that each neighbor of x_{d-2} with the exception of x_{d-3} is a leaf or a support vertex of degree two by Lemma 5.1. Assume that x_{d-2} has a neighbors as support vertices of degree two and b neighbors as leaves. Clearly $a + b \geq 2$. Let G' be the component of $G - x_{d-2}$ that contains no leaf-neighbor or support neighbor of x_{d-2}. By the inductive hypothesis, $\gamma_{oiR}(G') \leq 9n'/10$. Let f' be a $\gamma_{oiR}(G')$-function. Then we extend f' to an OIRDF for G by assigning 2 to x_{d-2}, 0 to its neighbors which are leaves or support vertices of degree two, and 1 to any other vertex. Since $a + b \geq 2$, we obtain that

$$\begin{aligned}\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 2 + a &\leq \frac{9}{10}(n - 2a - b - 1) + 2 + a \\ &= \frac{9n - 8a - 9b + 11}{10} < \frac{9}{10}n.\end{aligned}$$

We thus assume that $\deg(x_{d-2}) = 2$.

Assume that $d \geq 4$. Assume $\deg(x_{d-3}) \geq 3$. Then we may assume that each neighbor of x_{d-3} with the exception of x_{d-4} is a leaf, a support vertex of degree two, or a vertex of degree two that is adjacent to a support vertex of degree two. Assume that x_{d-3} has a neighbors as vertices of degree two that are adjacent to support vertices of degree two, b neighbors as support vertices of degree two, and c neighbors as leaves. Clearly $a + b + c \geq 2$. Let A be the set of all such neighbors of x_{d-3}. Let G' be the component of $G - x_{d-3}$ that contains no vertex of A. By the inductive hypothesis, $\gamma_{oiR}(G') \leq 9n'/10$. Let f' be a $\gamma_{oiR}(G')$-function. We extend f' to an OIRDF for G by assigning 2 to x_{d-3} and each support vertex at distance 2 from x_{d-3}, 1 to any leaf at distance two from x_{d-3} and 0 to any other vertex in $V(G) - V(G')$. Then

$$\begin{aligned}\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 2 + 2a + b &\leq \frac{9}{10}(n - 3a - 2b - c - 1) + 2 + 2a + b \\ &= \frac{9n - 7a - 8b - 9c + 11}{10} < \frac{9}{10}n,\end{aligned}$$

since $a+b+c \geq 2$. Thus assume that $\deg(x_{d-3}) = 2$. Let $G' = G - \{x_d, x_{d-1}, x_{d-2}, x_{d-3}\}$. By the inductive hypothesis, $\gamma_{oiR}(G') \leq 9n'/10$. Let f' be a $\gamma_{oiR}(G')$-function. Then we extend f' to an OIRDF for G by assigning 2 to x_{d-1}, 1 from x_{d-3}, and 0 to x_d and x_{d-2}. Then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 3 \leq \frac{9}{10}(n-4) + 3 < \frac{9}{10}n$.

Thus assume that $d = 3$. Let $G' = G - \{x, x_1, x_2, x_3\}$. If G' is connected, then $G' \neq K_3$, since G is a bicyclic graph. By Theorem 2.3, $\gamma_{oiR}(G') \leq 8n'/9$. Let f' be a $\gamma_{oiR}(G')$-function. Then we extend f' to an OIRDF for G by assigning 2 to x_2, 1 to x, and 0 to x_1 and x_3. Then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 3 \leq \frac{8}{9}(n-4) + 3 < \frac{9}{10}n$. Next assume that G' is disconnected. If G' has no K_3-components, then as before we find that $\gamma_{oiR}(G) < \frac{9}{10}n$. Thus assume that G' has some K_3-components. If G' has two K_3-components, then $|C_1| = |C_2| = 3$, $s = 2$ and $P = v_1 x u_1$. Then $n = 10$ and $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. Thus assume that G' has precisely one K_3-component. Without loss of generality, assume that C_1 is such component. Let $G' = G - \{x, x_1, x_2, x_3, v_1, v_2, v_3\}$. By Theorem 2.3, $\gamma_{oiR}(G') \leq 8n'/9$. Let f' be a $\gamma_{oiR}(G')$-function. Then we extend f' to an OIRDF for G by assigning 2 to x, 1 to x_2, x_3, v_2, v_3, and 0 to x_1 and v_1. Then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 6 \leq \frac{8}{9}(n-7) + 6 < \frac{9}{10}n$. □

Lemma 5.3 *If x is a vertex in $P \cup C_1 \cup C_2$ with $x \notin \{v_1, u_1\}$ and $\deg(x) \geq 4$, then $\gamma_{oiR}(G) < \frac{9}{10}n$.*

Proof. Assume that x is a vertex in $P \cup C_1 \cup C_2$ and $x \notin \{v_1, u_1\}$ and $\deg(x) \geq 4$. By Lemmas 5.1 and 5.2, each neighbor of x outside $P \cup C_1 \cup C_2$ is a leaf or a support vertex of degree two.

Assume that x has a neighbors as support vertices of degree two and b neighbors as leaves. Clearly $a + b \geq 2$. Let G' be the component of $G - x$ that contains no leaf-neighbor or support neighbor of x. If G' has no K_3-components, then by Theorem 2.3, $\gamma_{oiR}(G') \leq 8n'/9$. Let f' be a $\gamma_{oiR}(G')$-function. Then we extend f' to an OIRDF for G by assigning 2 to x, 0 to its neighbors which are leaves or support vertices of degree two, and 1 to any other vertex. Since $a + b \geq 2$, we obtain that $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 2 + a \leq \frac{8}{9}(n - 2a - b - 1) + 2 + a < \frac{9}{10}n$. Thus assume that G' has some K_3-components. If G' has two K_3-components, then $|C_1| = |C_2| = 3$, $s = 2$ and $P = v_1 x u_1$. Then $n = 2a + b + 7$ and $\gamma_{oiR}(G) = 6 + a < \frac{9}{10}n$. Thus assume that G' has precisely one K_3-component. Without loss of generality, assume that C_1 is such component. Let $G'' = G' - \{v_1, v_2, v_3\}$. By Theorem 2.3, $\gamma_{oiR}(G'') \leq 8n'/9$. Let f' be a $\gamma_{oiR}(G'')$-function. Then we extend f' to an OIRDF for G by assigning 2 to x, 1 to v_2, v_3 and the leaves at distance two from x, and 0 to each other vertex. Then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G'') + 4 + a \leq \frac{8}{9}(n - 2a - b - 4) + 4 + a < \frac{9}{10}n$. □

By Lemma 5.3 if x is a vertex in $P \cup C_1 \cup C_2$ and $x \notin \{v_1, u_1\}$, then we may assume that $\deg(x) \leq 3$. Similarly, we may assume that the following hold.

Lemma 5.4 $3 \leq \deg(v_1) \leq 4$ *and* $3 \leq \deg(u_1) \leq 4$.

From Lemmas 5.1 and 5.2, we may assume that for any vertex $x \in P \cup C_1 \cup C_2 - \{v_1, u_1\}$ with $\deg(x) = 3$, x is a support vertex or adjacent to a support vertex of degree two, and if $\deg(x) = 4$ for $x \in \{u_1, v_1\}$, then x is a support vertex or adjacent to a support vertex of degree two.

Lemma 5.5 *If $n > 10$, then $\gamma_{oiR}(G) < \frac{9}{10}n$.*

Proof. Assume that $n > 10$. Assume that there is a vertex $x \in C_1 \cup C_2$ such that $\deg(x) = 2$. Clearly, $x \notin \{v_1, u_1\}$. Let $G' = G-x$. By Theorem 2.3, $\gamma_{oiR}(G') \leq 8n'/9$. Let f' be a $\gamma_{oiR}(G')$-function. Then we extend f' to an OIRDF for G by assigning 1 to x. Then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 1 \leq \frac{8}{9}(n-1) + 1 < \frac{9}{10}n$. Thus assume that each vertex of $C_1 \cup C_2$ with the exception of v_1 and u_1 are of degree 3.

Assume that $|C_1| = 3$. Let G' be the component of $G - v_1$ containing u_2. Then G' is a tree or a unicyclic graph. If G' is a tree then by Theorem 1.1, $\gamma_{oiR}(G') < \frac{5}{6}n'$ and if G' is a unicyclic graph, then by Theorem 2.3, $\gamma_{oiR}(G') < \frac{8}{9}n'$. We extend any $\gamma_{oiR}(G')$-function to an OIRDF for G by assigning 1 to v_1 and the remaining vertices are assigned values as follows. Assume that each vertex in $\{v_1, v_2, v_3\}$ is a support vertex. Then assign 2 to v_3, 0 to v_2 and the leaf-neighbor of v_3, and 1 to each other vertex. If G' is a tree, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 5 \leq \frac{5}{6}(n-6) + 5 < \frac{9}{10}n$, and if G' is a unicyclic graph, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 5 \leq \frac{8}{9}(n-6) + 5 < \frac{9}{10}n$. Thus, assume without loss of generality, that v_3 is not a support vertex, (note that the other possibilities are similar). Assume that both v_1 and v_2 are support vertices. Then we assign 2 to v_3, 0 to v_2 and the support-neighbor of v_3, and 1 to each other vertex. If G' is a tree, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G')+6 \leq \frac{5}{6}(n-7)+6 < \frac{9}{10}n$, and if G' is a unicyclic graph, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 6 \leq \frac{8}{9}(n-7) + 6 < \frac{9}{10}n$. Thus, assume that v_2 is not a support vertex. If $\deg(v_1) = 3$, then we we extend any $\gamma_{oiR}(G')$-function to an OIRDF for G by assigning 2 to v_3, 0 to v_2 and the support-neighbor of v_3, and 1 to each other vertex, and as before the result is valid no matter G' is a tree or a unicyclic graph. Thus, assume that $\deg(v_1) = 4$. If v_1 is a support vertex, then assign 2 to v_3, 0 to v_2 and the support-neighbor of v_3, and 1 to each other vertex. If G' is a tree, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 7 \leq \frac{5}{6}(n-8) + 7 < \frac{9}{10}n$, and if G' is a unicyclic graph, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 7 \leq \frac{8}{9}(n-8) + 7 < \frac{9}{10}n$. Thus assume that v_1 is not a support vertex. Then similarly, we find that if G' is a tree then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 8 \leq \frac{5}{6}(n-9) + 8 < \frac{9}{10}n$, and if G' is a unicyclic graph, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 8 \leq \frac{8}{9}(n-9) + 8 < \frac{9}{10}n$.

Next assume that $|C_1| > 3$. Let G' be the component of $G-\{v_2, v_3, v_4\}$ containing v_1. Then G' is a unicyclic graph, and by Theorem 2.3, $\gamma_{oiR}(G') < \frac{8}{9}n'$. Observe that $\deg(v_2) = \deg(v_3) = \deg(v_4) = 3$. We asign 2 to v_2, 0 to v_3 and the neighbor of v_2 outside C_1, and 1 to any other vertex. If none of v_2, v_3, v_4 is a support vertex, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 8 \leq \frac{8}{9}(n-9) + 8 < \frac{9}{10}n$. If precisely, one vertex in $\{v_2, v_3, v_4\}$ is a support vertex, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 7 \leq \frac{8}{9}(n-8) + 7 < \frac{9}{10}n$. If precisely, two vertices in $\{v_2, v_3, v_4\}$ are support vertices, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 6 \leq \frac{8}{9}(n-7) + 6 < \frac{9}{10}n$. Finally, if each vertex in $\{v_2, v_3, v_4\}$ is a support vertex, then $\gamma_{oiR}(G) \leq \gamma_{oiR}(G') + 5 \leq \frac{8}{9}(n-6) + 5 < \frac{9}{10}n$. □

Thus assume by Lemma 5.5 that $n = 10$. If $|C_1| \geq 4$, then assigning 2 to v_1, 0 to v_2, v_k and a neighbor of v_1 outside C_1, and 1 to each other vertex of G yields an

OIRDF for G, and thus $\gamma_{oiR}(G) \leq n-2 < \frac{9}{10}n$. Thus assume that $|C_1| = 3$ and likewise, $|C_2| = 3$.

Lemma 5.6 *If there is a vertex x with three neighbors x_1, x_2, x_3 such that $G[\{x, x_1, x_2, x_3\}] = K_{1,3}$, then $\gamma_{oiR}(G) < \frac{9}{10}n$.*

Proof. Assume that x is a vertex with three neighbors x_1, x_2, x_3 such that $G[\{x, x_1, x_2, x_3\}] = K_{1,3}$. Then assigning 2 to x, 0 to x_1, x_2, x_3, and 1 to each other vertex of G yields an OIRDF for G, and thus $\gamma_{oiR}(G) \leq n-2 < \frac{9}{10}n$. □

By Lemma 5.6, we assume that G is claw-free. We next continue according to values of $s = d(v_1, u_1)$.

Assume that $s = 0$. By Lemma 5.6, $\deg(v_1) = 4$. If $\deg(v_2) = \deg(v_3) = \deg(u_2) = \deg(u_3) = 3$, then we may assume, without loss of generality, that u_3 is not a support vertex, while v_2, v_3, u_2 are support vertices. Then $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. Thus, assume, without loss of generality, that $\deg(v_2) = 2$. Then $\deg(v_3) = \deg(u_2) = \deg(u_3) = 3$, since $n = 10$. If v_3 is a support vertex, then neither of u_2 and u_3 is a support vertex, and $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. Thus assume that v_3 is not a support vertex. Then one of u_2 and u_3 is a support vertex and the other one is not a support vertex, and we can see that $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$.

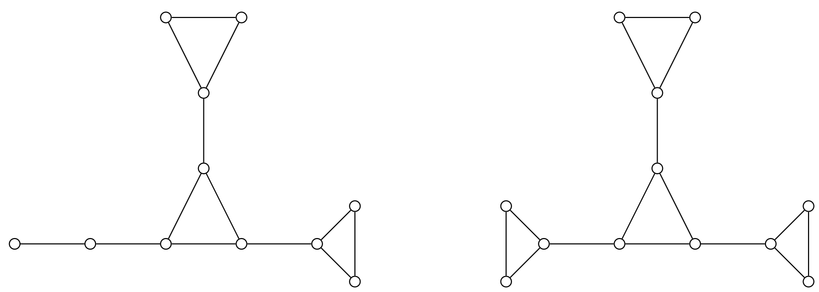
Thus $1 \leq s \leq 5$. Let $P : v_1 w_1 \ldots w_{s-1} u_1$. If $s = 5$, then by assigning 2 to w_1, u_2, 1 to w_3, v_2, v_3, u_2, and 0 to each other vertex we obtain that $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. If $s = 4$, then we may assume by Lemma 5.6 that precisely one vertex in $\{v_2, v_3, u_2, u_3\}$ is a support vertex. Assume that v_2 is such a vertex. Then by assigning 2 to w_2, v_2, 0 to v_3, w_1, w_3 and the leaf-neighbor of v_2, and 1 to each other vertex we obtain that $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. If $s = 3$, then by Lemma 5.6, either there is precisely one vertex of $\{v_2, v_3, u_2, u_3\}$ that is a adjacent to a support vertex of degree two or there are two vertices in $\{v_2, v_3, u_2, u_3\}$ that are support vertices. In each of this possibilities, we can see that $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. If $s = 2$, then by Lemma 5.6, either there are precisely three vertices of $\{v_2, v_3, u_2, u_3\}$ that are support vertices or there are two vertices x, y in $\{v_2, v_3, u_2, u_3\}$ such that x is a support vertex and y is adjacent to a support vertex of degree two. In each of these possibilities, we can see that $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$.

Finally, assume that $s = 1$. If $\deg(v_2) = \deg(v_3) = \deg(u_2) = \deg(u_3) = 3$, then $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. Thus assume, without loss of generality, that $\deg(u_2) = 2$. Assume that $\deg(u_3) = 3$. If u_3 is a support vertex, then we may assume that v_2 is a support vertex and v_3 is a vertex adjacent to a support vertex of degree two. Then $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. Thus, assume that u_3 is not a support vertex. Then u_3 is adjacent to a support vertex of degree two. Then either both v_2 and v_3 are support vertices or precisely one of them is adjacent to a support vertex of degree two. Then we observe that $\gamma_{oiR}(G) = 8 < \frac{9}{10}n$. We thus assume that $\deg(u_3) = 2$. Then each of v_2 and v_3 is adjacent to a support vertex of degree two. Consequently, $G = H_2$.

6 Concluding remarks

By Theorem 2.3, $\gamma_{oiR}(G) \leq 8n/9$ for a unicyclic graph $G \neq K_3$, and by Theorem 2.4, $\gamma_{oiR}(G) \leq 9n/10$ if G has two cycles. It is a good problem to investigate such a bound for cactus graphs. It seems that if a cactus graph G has three cycles then $\gamma_{oiR}(G) \leq 10n/11$, and if it has four cycles then $\gamma_{oiR}(G) \leq 11n/12$. Figure 2 illustrates two graphs achieving equality of the above proposed bounds. Furthermore, if the above bounds are correct then perhaps $\gamma_{oiR}(G) \leq (\frac{8+k}{9+k})n$ if G is a cactus graph with k cycles, but this earlier bound does not seem to be sharp. We propose these problems for researchers.

Conjecture 6.1 *If $G \neq K_3$ is a cactus graph of order n, then $\gamma_{oiR}(G) \leq 11n/12$, with equality if and only if $G = H_4$, where H_4 is the graph depicted in Figure* 2.



H_3 H_4

Figure 2. Cactus graphs with outer-independent Roman domination number $10n/11$ and $11n/12$. The graph H_3 has outer-independent Roman domination number 10 and the graph H_4 has outer-independent Roman domination number 11.

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