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Radius r extremal graphs of girth 5

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Abstract

For extremal graphs of girth at least 5, radius plays an important role in their structure, proofs of bounds on their sizes, and exact values of their sizes. All graphs in $\mathcal{F}$, the extremal graphs with girth at least 5, have radius 2 or 3 for orders greater than or equal to 5. However, not all radius 2 and 3 extremal graphs with girth at least 5, $\mathcal{F}^2 \cup \mathcal{F}^3$, are in $\mathcal{F}$. We prove that all graphs in $\mathcal{F}^2$, for $v \geq 5$, have girth 5 and diameter 2 or 3, and all graphs in $\mathcal{F}^3$, for $v \geq 9$, have girth 5 and diameter 3 or 4. We determine the exact sizes, or narrow bounds on the sizes, of the graphs in $\mathcal{F}^2$ and $\mathcal{F}^3$ for all orders up to 53. We enumerate, or set constructive lower bounds on the number of, the non-isomorphic graphs in $\mathcal{F}^2$ and $\mathcal{F}^3$ for these orders.

1 Introduction

The graphs under consideration are simple and undirected. We use these common terms and notation. The order and size of graph G are $v(G)$ and $e(G)$. The degree of vertex x is $d(x)$. The minimum and maximum degrees of all vertices in G are $\delta(G)$ and $\Delta(G)$, and $\bar{d}(G)$ is the average degree over all vertices in G. The induced subgraph on set X, where $X \subset V(G)$, is $\langle X \rangle$. The path and cycle on k vertices are P_k and C_k, and the girth of G is $g(G)$. The *eccentricity* of vertex x, $\mathrm{ecc}(x)$, is the maximum distance from x to any other vertex in G, and $r(G)$, the *radius* of G, is the minimum eccentricity over all $x \in V(G)$. The *center* of G, $\mathrm{cent}(G)$, is the set of vertices having minimal eccentricity; that is, $x \in \mathrm{cent}(G)$ if and only if $\mathrm{ecc}(x) = r(G)$.

In 1975 Erdős [8] described the problem of determining, for a given order v, $f(v)$ the maximum number of edges in a graph of girth at least 5. This is an example of a type of extremal graph problem called the forbidden subgraph problem. Given

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a set of graphs, Γ, determine the graphs $\mathrm{ex}(v;\Gamma)$ which have order v, maximal size, and do not have any subgraphs isomorphic to a graph in Γ. For convenience, in this paper we apply the term *extremal* specifically to graphs of girth at least 5, that is, graphs in $\mathrm{ex}(v;\{C_3,C_4\})$.

Extremal graphs with a minimum girth, and theoretical results on them, have been applied to problems in information theory, including error correction capability of parity-check codes [6], and in coding theory and cryptography [18]. Applications to the analysis of local majorities and coalitions have implications for distributed computing and voter polling [17]. And, in economics, applications include comparative analysis of the benefits of social welfare functions [12].

In this paper we study extremal radius constrained graphs with girth at least 5. There are two motivations for this. First, in the study of extremal graphs with minimum girth, radius plays a part in proofs of exact values, as we illustrate in Section 3. This merits further study of the structure of graphs for a given radius. Second, minimum girth extremal graphs represent a balance between connectivity and density that makes them potentially useful for the design of networks. The radius constrained graphs provide alternative designs with similar trade-offs, and radius is a useful measure for the distribution of resources in a network.

Since Erdős posed the problem of extremal graphs, theoretical bounds on $f(v)$ have been derived [1, 5, 10], and exact values for v up to 53 have been determined [2, 4, 7, 10, 11]. The non-isomorphic extremal graphs have been enumerated up to order 52 [2, 10, 11], and, using heuristic search and analytic methods, lower bounds for $f(v)$ have been found for $54 \le v \le 200$ [2, 11, 15].

Two-level trees play an important role in the structure of extremal graphs. We use the notation initiated in [10, 11] to describe these trees. A *star*, $S_{m[n_1,n_2,\ldots,n_m]}$, is a tree with a degree m vertex at the root, the m children of the root, and child i, $1 \le i \le m$, having n_i children. An *(m,n)-star*, $S_{m,n}$, is a special case of a star where every child of the root has n children. An *augmented star* is a star plus vertices external to it. It is denoted as a star with the number of external vertices appended. For example, $S_{m[n_1,n_2,\ldots,n_m]k}$ or $S_{m,n,k}$ are stars augmented with k vertices that are external to the star. Note that if an augmented star S spans G, there can be an augmented star T that also spans G, where S and T are non-isomorphic. A *branch* of a star S consists of a child of the root of S, and the leaves in S adjacent to that child. A *k-branch* of S has k leaves in S.

The presence of these stars in extremal graphs is the basis for many of the cited results. The results often depend on whether or not there are stars that span extremal graphs of a given order. When there is a star that spans an extremal graph G, then the radius of G is 2. If G has no spanning star, then the radius is at least 3, and only augmented stars span G. These are the observations that motivate the study of radius r extremal graphs. Graph G is a *radius r extremal graph* if $r(G) = r$, $g(G) \ge 5$, and $e(G)$ is maximal.

We use the following notation to describe extremal and radius r extremal graphs. The size of an extremal graph of order v is $f(v)$. The set of all extremal graphs is $\mathcal{F}$, and $\mathcal{F}_v$ is the set of graphs in $\mathcal{F}$ of order v. The number of extremal graphs of order v, $|\mathcal{F}_v|$, is denoted $F(v)$. The size of a radius r extremal graph of order v is $f_r(v)$. For some orders v and radii r, $f_r(v) < f(v)$. For example, as shown in Section 3, $f_3(7) < f(7)$; the additional constraint on the radius limits the graph size. The set of radius r extremal graphs of order v is $\mathcal{F}_v^r$. The number of radius r extremal graphs of order v, $|\mathcal{F}_v^r|$, is $F_r(v)$.

In the next section we present results on the structure of graphs in $\mathcal{F}^2$ and $\mathcal{F}^3$. Section 3 provides exact values or narrow bounds on the values of $f_2(v), f_3(v), \mathcal{F}_v^2$, and $\mathcal{F}_v^3$ for $v \leq 53$. And in the conclusion we offer conjectures and directions for future work.

2 Theoretical results

We observe that all extremal graphs are connected. This is seen by noting that, if a graph has more than one connected component, a pair of them can be bridged with an additional edge that does not create a cycle, and therefore the original graph cannot be extremal.

We now prove that all extremal graphs, large enough to have a cycle, have radius 2 or 3. This is the simple observation that led to this study.

Proposition 2.1 *Let $v \geq 5$. Then for every $G \in \mathcal{F}_v$, $2 \leq r(G) \leq 3$.*

Proof. If $r(G) < 2$ and $v \geq 5$, then, since G is connected, it is a vertex x with $v - 1$ neighbors, and no edges between neighbors of x. Thus, $e = v - 1$. Since $g(C_v) = e(C_v) = v$, G is not extremal.

If $r(G) > 3$, then $x \in \text{cent}(G)$ is at least distance 4 from some vertex y. Since $g(G \cup \{(x, y)\}) \geq 5$, G is not extremal. Therefore $2 \leq r(G) \leq 3$, and examples with both radius 2 and radius 3 are known to be in $\mathcal{F}$. □

The authors of [10] proved bounds on the diameter of extremal graphs. We restate their proposition and proof here, with the addition that the bounds apply to radius 2 extremal graphs as well, even when $f_2(v) < f(v)$.

Proposition 2.2 *If $G \in \mathcal{F}_v \cup \mathcal{F}_v^2$, then*

1. *the diameter of G is at most 3;*
2. *if $d(x) = \delta(G) = 1$, then the graph $G - \{x\}$ has diameter at most 2.*

Proof. For x and y in $V(G)$, at least distance 4 apart, let $V(H) = V(G)$ and $E(H) = E(G) \cup \{(x, y)\}$. Since $g(H) \geq 5$ and $e(H) = e(G) + 1$, G is neither extremal nor radius 2 extremal, thus proving part 1.

For part 2, let pendant vertex x be adjacent to x', and let y and z be two vertices in $V(G) - \{x\}$ where y and z are at least distance 3 apart. Let $V(H) = V(G)$ and $E(H) = (E(G) - \{(x, x')\}) \cup \{(x, y), (x, z)\}$. Again, since $g(H) \geq 5$ and $e(H) = e(G) + 1$, G is neither extremal nor radius 2 extremal, thus proving part 2. □

The constructions in Proposition 2.2 potentially reduce the radius of a graph, and therefore the proposition does not hold for radius 3 extremal graphs. Many of the known graphs in $\mathcal{F}_v^3$, where $f_3(v) < f(v)$, have diameter 4. This is the upper bound on the diameter of a graph in $\mathcal{F}_v^3$, as stated in Proposition 2.3.

Proposition 2.3 *If $G \in \mathcal{F}_v^3$ then the diameter of G is at most 4.*

Proof. If x and y in $V(G)$ are at least distance 5 apart, then let $V(H) = V(G)$ and $E(H) = E(G) \cup \{(x, y)\}$. Since adding an edge does not increase the radius, $r(H) \leq 3$, and the girth of H is still at least 5. Also, $r(H) \geq 3$ since, if $r(H) = 2$, there is vertex $c \in \mathrm{cent}(H)$ where $\mathrm{ecc}(c) > 2$ in G, and $\mathrm{ecc}(c) = 2$ in H. But this would contradict that x and y are at least distance 5 apart in G. This is true whether such a vertex c is external to the path $(x, \ldots, y)$ in G, illustrated as an example in Figure 1A, or a vertex in the path, illustrated as an example in Figure 1B. The dashed edge (x, y) is added to construct H. The other dashed edges, and dashed vertex, are examples of what would be present in G to allow the radius of H to become 2 by the addition of (x, y). Thus, $g(H) \geq 5$, $r(H) = 3$, and $v(H) = v(G)+1$, which contradicts that $G \in \mathcal{F}_v^3$. Therefore, the diameter of any graph in $\mathcal{F}^3$ is at most 4. □

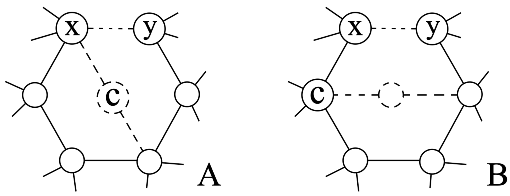

Figure 1: For Proposition 2.3 showing if $G \in \mathcal{F}^3$, its diameter is at most 4

The authors of [10] observed that the minimum degree of any extremal graph G, of order v, must be at least the difference between $f(v)$ and $f(v-1)$; otherwise, removing a vertex of the minimum degree from G would create a graph with girth at least 5 and order $v-1$ where $e > f(v-1)$. They stated this formally in the following proposition.

Proposition 2.4 *For any graph with girth at least 5 and $v > 1$, $\delta \geq e - f(v-1)$.*

This was generalized in [11] to a lower bound on the number of edges that can be incident on any set of k vertices in a graph with girth at least 5.

Proposition 2.5 *For any k vertices, $x_1, x_2, \ldots, x_k$ in $V(G)$, where $g(G) \geq 5$,*

$$\sum_{i=1}^{k} d(x_i) - |E(\langle x_1, x_2, \ldots, x_k \rangle)| \geq e - f(v-k).$$

We note that the inequalities in Propositions 2.4 and 2.5 do not necessarily hold when the right sides of the equations are restricted to radius r graphs. For graph

G, with girth at least 5 and radius equal to 3, it is not necessarily the case that $\delta \geq e - f_3(v-1)$. For example, $f_3(16) = 28$, $f_3(15) = 25$, and there is a graph $G \in \mathcal{F}^3_{16}$ with $\delta = 2$. In that case, $\delta(G) < e(G) - f_3(v-1)$. This is possible because $f_3(16) = f(16)$, but $f_3(15) = f(15) - 1$. Similarly, for radius 2, $f_2(20) = 41$, $f_2(19) = 37$, and there exists $G \in \mathcal{F}^2_{20}$ with $\delta = 3$.

The authors of [10] describe the presence of stars, rooted depth 2 trees, embedded in extremal graphs. We note that $v(S_{m,n}) = mn + m + 1$. Since all extremal and radius r extremal graphs are connected, every vertex is the root of a tree that spans the graph. It follows that any vertex in an extremal graph, with degree Δ, is the root of a tree where the other vertices in the tree have degree at least δ. Therefore we have this simple proposition. It was described in [10] for extremal graphs. Here we apply it to radius 2 and radius 3 extremal graphs as well.

Proposition 2.6 *For $G \in \mathcal{F}^2_v \cup \mathcal{F}^3_v$, where $v \geq 5$,*

1. *graph G contains star $S_{\Delta,\delta-1}$;*
2. *if $G \in \mathcal{F}^2_v$ then $v \geq \Delta\delta + 1$;*
3. *if $G \in \mathcal{F}^3_v$ then $v \geq \Delta\delta + 2$.*

The bound on the number of vertices in a radius 3 extremal graph is one higher since, if $r(G) = 3, G$ must have at least one vertex external to any embedded star.

In determining δ and Δ, it is useful to observe that the average degree of the vertices in a graph bounds δ from above and Δ from below.

Proposition 2.7 *For any graph, $\delta \leq 2e/v \leq \Delta$.*

Since $\mathcal{F}$ is defined without regard to the radius of its graphs and, by Proposition 2.1, every extremal graph of order at least 5 has radius 2 or 3, we have the following proposition.

Proposition 2.8 *For $v \geq 6, f(v) = \max(f_2(v), f_3(v))$.*

We note that $f_2(v)$ is undefined for $v < 4$ since there are no radius 2 extremal graphs for such small orders. Similarly, $f_3(v)$ is undefined for $v < 6$. We also note that, for various orders v, $f_2(v) = f_3(v) = f(v)$, $f_2(v) < f(v)$, or $f_3(v) < f(v)$. For example:

1. $f_2(11) = f_3(11) = f(11) = 16$;
2. $f_2(19) = 37$ and $f_3(19) = f(19) = 38$;
3. $f_3(9) = 11$ and $f_2(9) = f(9) = 12$.

This leads us to investigate whether there are v and $r, r > 3$, where $f_r(v) > f_3(v)$ or $f_r(v) > f_2(v)$. Theorem 2.9 states that this is not the case. Note that $f_r(v)$ is not defined for $v < 2r$.

Theorem 2.9 *For $v \geq 2r$,*

1. $f_r(v) < f_2(v)$ *if* $r \geq 4$;
2. $f_4(v) \leq f_3(v)$;
3. $f_r(v) < f_3(v)$ *if* $r \geq 5$.

Proof. If G is radius r extremal, and $r > 3$, then for $x \in \text{cent}(G)$, there exists $y \in V(G)$ such that $\text{dist}(x, y) = r$. Let $G' = G \cup \{(x, y)\}$. The additional edge does not create a cycle shorter than C_5, and further, $v(G') = v(G), e(G') = e(G) + 1$, and $2 \leq r(G') \leq r(G)$.

An application of the construction either produces graph G' with $r(G') < r$, or reduces the number of vertices that are distance r from a center vertex. Repeat the construction recursively until $r(G') = 2$. It is the case that $v(G') = v(G), e(G') > e(G), g(G') \geq 5$, and $r(G') = 2$, thus proving part 1.

To prove part 2, consider a path (x, x_1, x_2, x_3, x_4) in G where $x \in \text{cent}(G)$, and $\text{dist}(x, x_4) = 4$. We can reduce the number of vertices that are distance 4 from x by adding edge (x, x_4) and removing edge (x_3, x_4). That is, let $G' = G \cup \{(x, x_4)\} - \{(x_3, x_4)\}$. The new graph G' has the same order and size as G and still has girth of at least 5. If $r(G') = 2$, that would contradict that $r(G) = 4$, so $3 \leq r(G') \leq 4$. The construction has reduced by at least one the number of vertices that are distance 4 from x. Repeat the construction recursively on G', with the same vertex x, until the eccentricity of x in G' is 3. Since $v(G') = v(G)$, $e(G') = e(G)$, $g(G') \geq 5$, and $r(G') = 3$, part 2 is proven.

To prove part 3, apply the first construction recursively until $3 \leq r(G') \leq 4$. If $r(G') = 4$ apply the second construction recursively until $r(G') = 3$. It is the case that $v(G') = v(G)$, $e(G') > e(G)$, $g(G') \geq 5$, and $r(G') = 3$, thus proving part 3. □

We note that in generating and examining radius r extremal graphs for orders up to 53, we have not discovered a graph G where $g(G) \geq 5, r(G) = 4$, and $e(G) = f_3(v)$. We conjecture that $f_4(v) < f_3(v)$ for $v \geq 8$.

It is obvious that $f(v) > f(v-1)$ by observing that a pendant vertex can be added to G in $\mathcal{F}_{v-1}$. Similarly, $f_2(v) > f_2(v-1)$ and $f_3(v) > f_3(v-1)$, if a pendant vertex is added adjacent to a vertex in $\text{cent}(G)$ for G in $\mathcal{F}^2$ or $\mathcal{F}^3$. However, it is not as obvious that $f_3(v) > f(v-1)$ for all v since $f_3(v) < f_2(v)$ for some values of v. To prove that this is the case, we must show that $f_3(v) > f_2(v-1)$ where $f_2(v-1) = f(v-1)$. If $f_2(v-1) < f(v-1)$, it follows from Proposition 2.8 that $f_3(v-1) = f(v-1)$ and therefore $f_3(v) > f_3(v-1) > f_2(v-1)$.

Theorem 2.10 *For $v \geq 6$, $f_3(v) > f(v-1)$.*

Proof. It only remains to be proven that $f_3(v) > f_2(v-1)$ where $f_2(v-1) = f(v-1)$. The authors of [11] demonstrated that for all extremal graphs G, with $v \geq 7$, $g(G) = 5$. This, and by inspection of the graphs in Figure 5, establishes that if $G \in \mathcal{F}_v^2, f_2(v) = f(v)$, and $v \geq 5$, then G contains a C_5.

Let (y, z) be any edge in a C_5 in G, where $G \in \mathcal{F}^2_{v-1} \cap \mathcal{F}_{v-1}$ and $v \geq 6$. Consider graph H, as illustrated in Figure 2, where $V(H) = V(G) \cup \{x\}$, and $E(H) = E(G) \cup \{(x, y), (x, z)\} - \{(y, z)\}$. Graph H is such that $v(H) = v(G)+1$ and $e(H) = e(G)+1$. No cycle has contracted by the construction, so $g(H) \geq 5$.

For any vertex c, in the center of G, but not in the C_5 used in the construction, c was adjacent to at most one vertex in the C_5. Otherwise, the girth of G was less than 5. Therefore, c is distance 3 from at least one vertex in the constructed C_6. Similarly, for any center c that was in the C_5 in G, it is now distance 3 from one vertex in the constructed C_6. Thus, $r(H) = 3$ and $f_3(v) \geq f_2(v-1)+1$, and therefore $f_3(v) > f(v - 1)$. □

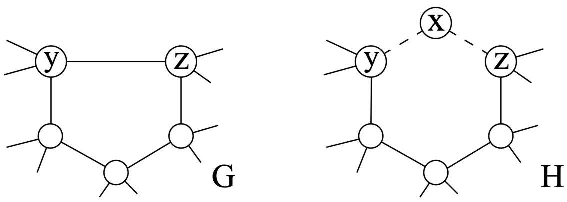

Figure 2: Construct an order v radius 3 $\{C_3, C_4\}$-free graph from a graph in $\mathcal{F}^2_{v-1}$

The authors of [11] proved that for all extremal graphs G, with $v \geq 7, g(G) = 5$. This was a corollary of the stronger result that, for x, a vertex in extremal graph G with $v \geq 5$,

1. if $d(x) = 2$, then G has a 5-cycle or 6-cycle that contains x;
2. if $d(x) \geq 3$, then G has a 5-cycle that contains x.

Where $\mathcal{F}_v$ and $\mathcal{F}^2_v$ have been determined, all degree 2 vertices are in a C_5, with the exceptions of $v \in \{6, 11, 51\}$. These orders are one more than the orders of the Moore graphs, and $f(v) = f(v-1)+1$. This is relevant to the presence of such degree 2 vertices, as shown in the following theorem which strengthens the result from [11]. This theorem uses the same construction from Theorem 2.10, but in reverse.

Theorem 2.11

1. *For all* $G \in \mathcal{F}_v \cup \mathcal{F}^2_v$, *where* $v \geq 5$ *and* $v \notin \{6, 11, 51, \text{and possibly } 3251\}$, *every* $x \in V(G)$ *is in a* C_5 *in* G.
2. *For all* $G \in \mathcal{F}_v \cup \mathcal{F}^2_v$, *where* $v \in \{6, 11, 51, \text{and possibly } 3251\}$, *and for every* $x \in V(G)$,

   (a) *if* $d(x) = 2$, *then* x *is in a* C_5 *or* C_6 *in* G;

   (b) *if* $d(x) \geq 3$, *then* x *is in a* C_5 *in* G.

Proof. From [10], G does not contain a pendant vertex unless $v(G) \in \{6, 11, 51,$ and possibly $3251\}$.

The authors of [11] provided a construction showing that if graph G with $g(G) \geq 5$ contained vertex x, of degree at least 3, that was not in a C_5, H can be constructed where $v(H) = v(G)$, $g(H) \geq 5$, and $e(H) = e(G) + 1$, thus proving that G is not extremal. The construction does not rely on G being extremal. Thus the proof holds as well for $G \in \mathcal{F}_v^2$ where $f_2(v) < f(v)$. Thus, for $G \in \mathcal{F}_v \cup \mathcal{F}_v^2$, every $x \in V(G)$, where $d(x) \geq 3$, is in a C_5. The construction potentially reduces the radius, and therefore does not necessarily apply to graphs in $\mathcal{F}_v^3$.

The construction in [11] also proved that if $x \in V(G)$ and $d(x) = 2$, then x is in either a C_5 or C_6 in G. Assume that the smallest cycle containing x is C_6. Then x is adjacent to vertices y and z as in graph G in Figure 3. Let $V(H) = V(G) - \{x\}$ and $E(H) = E(G) \cup \{(y, z)\}$, as also shown in Figure 3.

Since $g(G) = 5$, and the smallest cycle containing x in G is C_6, $g(H) = 5$. Also, since $G \in \mathcal{F}_v \cup \mathcal{F}_v^2, v(H) = v(G) - 1$, and $e(H) = e(G) - 1$, then $H \in \mathcal{F}_{v-1} \cup \mathcal{F}_{v-1}^2$. Graph H, plus a pendant vertex, is also in $\mathcal{F}_v \cup \mathcal{F}_v^2$, which, from [10], proves that H is a Moore graph. Thus, if the smallest cycle containing a degree 2 vertex in G is C_6, then $v(G) \in \{6, 11, 51,$ and possibly $3251\}$. Otherwise, any degree 2 vertex in G is in a C_5. $\square$

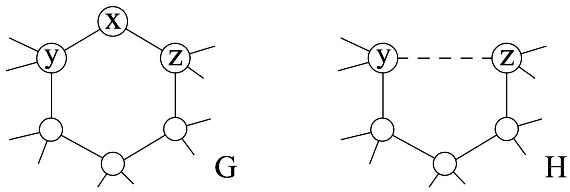

Figure 3: Construct a Moore graph by removing a degree 2 vertex in a 6-cycle

Lazebnik and Wang [13] used a construction to prove the general result that, if $n \geq 3$, $G \in \mathrm{ex}(v; \{C_3, C_4, \ldots, C_n\})$, and $\Delta(G) \geq n$, then $g(G) = n + 1$. We demonstrate that it applies to the case where $G \in \mathcal{F}^3$, even when $f_3(v) < f(v)$. And further, every vertex in G with degree at least 4 is in a 5-cycle. Our result relies on any vertex having at most one pendant neighbor.

Lemma 2.12 *For $G \in \mathcal{F}^3$, and $x \in V(G)$, x has at most one pendant neighbor.*

Proof. Suppose $r(G) = 3$ and assume $x \in V(G)$ has at least two pendant neighbors, x_1 and x_2. If we let $H = G - \{x_2\}$, then $v(H) = v(G) - 1$ and $e(H) = e(G) - 1$. By Theorem 2.10, $H \in \mathcal{F}_{v-1}$.

By Proposition 2.2, $H - \{x_1\}$ is a Moore graph. Therefore the eccentricity of x is 2 in both G and H. Since x_1 and x_2 are both adjacent to x in G, $r(G) = 2$. This contradicts that $x \in V(G)$, where $G \in \mathcal{F}^3$, can have two pendant neighbors. $\square$

We note that graph 7a in Figure 6 is the only known graph in $\mathcal{F}^3$ with a pendant vertex. We conjecture that all radius 3 extremal graphs, with $v \geq 8$, have $\delta \geq 2$.

Theorem 2.13 *If $G \in \mathcal{F}^3$ and $x \in V(G)$, then if $d(x) \geq 4$, x is in a C_5 in G.*

Proof. Assume that the smallest cycle in G containing x, with $d(x) \geq 4$, is C_k where $k > 5$. Vertex x is the root of star S where x has $d(x)$ children labeled $x_1, x_2, \ldots, x_{d(x)}$. The leaves of branch $b_i, 1 \leq i \leq d(x_i)$, of S include $N(x_i) - \{x\}$. If x has a pendant neighbor, let it be x_3. By Lemma 2.12, x has at most one pendant neighbor.

All remaining vertices in $V(G)$ are external to S. The case where $d(x) = 4$ is illustrated as graph G in Figure 4. Since x is not in a C_k, $k \leq 5$, there are no edges in $E(G)$ incident only on leaves of S. Each edge in $E(G) - E(S)$ is incident only on a pair of vertices external to S, or incident on a leaf of S and a vertex external to S.

Let H be constructed such that $V(H) = V(G)$ and

$$E(H) = E(G) \cup \{(x_2, x_3), (x_3, x_4), \ldots, (x_{d(x)}, x_1)\} - \{(x, x_3), (x, x_4), \ldots, (x, x_{d(x)})\}.$$

In the example in Figure 4, where $d(x) = 4$, the deleted edges are shown as dashed lines in graph G, and the added edges are shown as dashed lines in graph H. Since x is not in a C_5 in G, the added edges in H will not complete a C_3 nor a C_4, and therefore $g(H) \geq 5$. Note that $v(H) = v(G)$ and

$$e(H) = e(G) + (d(x) - 1) - (d(x) - 2) = e(G) + 1.$$

Since every vertex x in H has eccentricity greater than 2, then $r(H) \geq 3$. This is verified by observing that each of these distances is greater than 2:

1. $\text{dist}(x, y)$, where y is any vertex external to the star;
2. $\text{dist}(x_1, y)$, where y is any leaf in $b_i, 2 \leq i < d(x)$;
3. $\text{dist}(x_2, y)$, where y is any leaf in b_i, $i = 1$ or $4 \leq i \leq d(x)$;
4. $\text{dist}(x_i, y)$, where $3 \leq i < d(x)$ and y is any leaf in b_1;
5. $\text{dist}(x_{d(x)}, y)$, where y is any leaf in b_2.

If $r(H) = 3$, then $G \notin \mathcal{F}^3$, and x must be in a C_5 in G. If $r(H) > 3$, then by Theorem 2.9 H' can be constructed from H such that $v(H') = v(G), e(H') > e(G), g(H') \geq 5$, and $r(H') = 3$. Again, contradicting that x is not in a C_5 in G. □

For all known graphs $G \in \mathcal{F}^3$, with only two exceptions, if $x \in V(G)$ and $d(x) = 3$, x is in a C_5 in G. The two exceptions are graphs 7a and 8b in Figure 6. We conjecture that for all $G \in \mathcal{F}^3_v$ where $v \geq 9$, if $d(x) = 3$, then x is in a C_5 in G.

Corollary 2.14 *For $v \geq 9$, if $G \in \mathcal{F}^3_v$, then $g(G) = 5$.*

Proof. For $9 \leq v \leq 12$, it is true by direct inspection of the graphs. For $13 \leq v \leq 53$, from Table 1, $\Delta \geq \lceil \bar{d} \rceil = \lceil 2f_3(v)/v \rceil \geq 4$, and by Theorem 2.13, $g(G) = 5$.

For $v > 53$, it remains to be shown that if $G \in \mathcal{F}^3_v$, then $\Delta \geq 4$. Construct G, with $v > 53$ and $e \geq 2v$, by selecting a set of graphs, $\{G_1, G_2, \ldots, G_n\}, G_i \in \mathcal{F}^3_v, 19 \leq v \leq 53$, such that $\sum_{i=1}^{n} v(G_i) - (n-1) = v$. Overlap the set of graphs at a single shared vertex x where, from each graph G_i, some $x_i \in \text{cent}(G_i)$ is to be

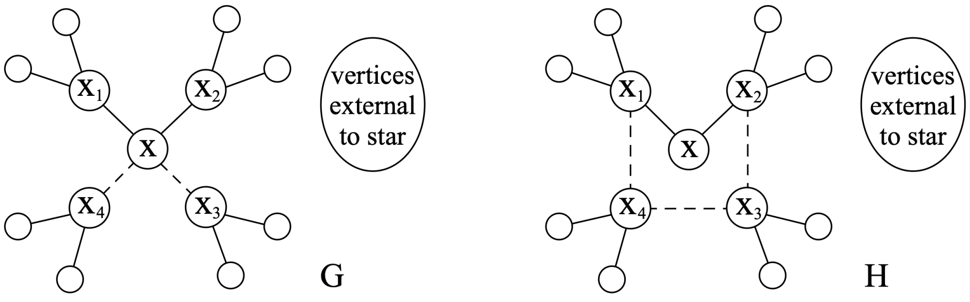

Figure 4: Example for Theorem 2.13 with $d(x) = 4$

the shared vertex in $V(G)$. The eccentricity of x is 3, and therefore the radius of G is 3. Also, the girth of G is 5. Since $e(G) \geq 2v(G)$ then $\bar{d}(G) \geq 4$. Therefore, $f_3(v) \geq e(G) \geq 2v, v > 53$, thus proving the corollary.

As an example, construct G, with $v(G) = 58$ and $e(G) = 120$, by having three copies of a graph $G_{20} \in \mathcal{F}^3_{20}$ share a single vertex $x \in \text{cent}(G_{20})$. The average degree in G is greater than 4, and the radius of G is 3. $\square$

3 Values and bounds for $f_2(v)$, $F_2(v)$, $f_3(v)$, and $F_3(v)$

We illustrate the radius 2 and radius 3 extremal graphs, up to order 10, in Figures 5 and 6. For these small orders the radius 2 extremal graphs are also extremal. Only the order 6 radius 3 extremal graph is also extremal. There are no radius 2 extremal graphs for orders less than 4, nor radius 3 extremal graphs for orders less than 6.

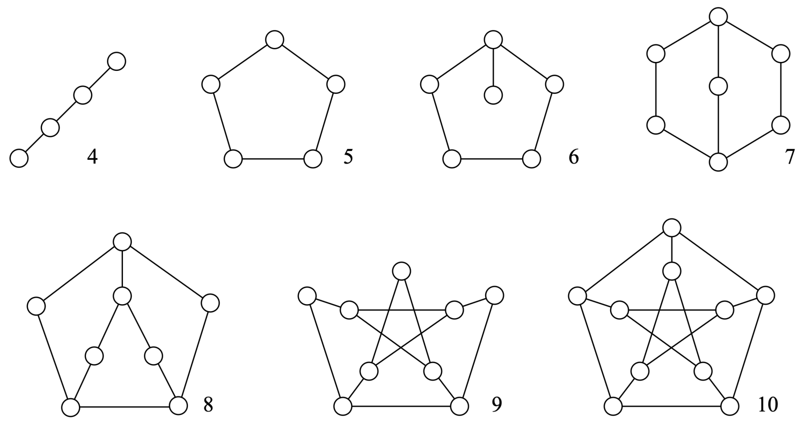

Figure 5: Radius 2 extremal graphs of orders 4 through 10

Table 1 lists values and bounds for $f_2(v)$, $f_3(v)$, $F_2(v)$, and $F_3(v)$. The values where $f_r(v) = f(v)$ were determined by simple algorithmic examination of the known extremal graphs obtained from Afzaly and McKay [3]. Their collection also includes graphs that establish constructive lower bounds for several orders larger than 53 which they generated with heuristic methods [2]. We include some of these values in the table. All known graphs in $\mathcal{F}_v^2 \cup \mathcal{F}_v^3, v \leq 52$, are available at [9].

Establishing the values for $f_r(v)$, where $f_r(v) = f(v) - 1$, it was sufficient to observe that there were no graphs in $\mathcal{F}_v$ with radius r, and then to generate by computer radius r graphs with $e = f(v) - 1$. In some cases we could not find exact values for $f_r(v)$, but do provide computational lower bounds and analytical upper bounds.

The computational methods combine hill climbing with backtracking. The hill climbing algorithm starts with v isolated vertices, and then randomly selects non-edges to be added. If the proposed edge does not complete a short cycle (a C_3 or C_4) the edge is added to the graph; this constitutes an uphill step. If the edge completes a single short cycle, then the edge is added and another edge from the short cycle is removed to break the cycle; this constitutes a sideways step. If the edge completes more than one short cycle, it is rejected as it would be a downhill step.

The hill climbing algorithm was both fast and effective for finding extremal graphs and radius 2 extremal graphs up to order 50. However, it was less effective on its own for radius 3 graphs for orders greater than about 45. For those graphs, improved results were obtained by following hill climbing with removing k edges, and using backtracking to add more than k edges, while maintaining radius 3 and girth 5.

For enumerating elements of $\mathcal{F}_v^r$ when $f_r(v) < f(v)$, it is necessary to identify isomorphs. The method we used for determining isomorphism is typical of that described by McKay and Piperno [14], which relies on canonically labeling each graph G. The tractability of the method depends on refining a partition of the vertices of G in order to reduce the number of permutations to test during the labeling; each vertex in G only needs to be mapped onto each of the other vertices in its own block in a partition. A finer partition requires fewer permutations to test for generating the canonical labeling. We used the partitioning method described in [11]. Each vertex x in $V(G)$ is colored with the number of 5-cycles in G that contain x. This generally produced a finer partition than coloring x with its degree index.

The coloring also provides a graph invariant, the C_5-*sequence* of a graph. The C_5-sequence, $A(G)$, is $[a_0, a_1, \ldots, a_{\binom{v}{5}}]$, where a_i is the number of vertices in $V(G)$ that are in i 5-cycles in G. Since most of the terms are zero, we only write the non-zero elements of the sequence, annotating them with their index. For example, one of the graphs in $\mathcal{F}_{12}^2$ has the C_5-sequence $[1{:}2, 6{:}7, 7{:}3]$ indicating two vertices are in a single 5-cycle, seven are in six 5-cycles, and three are in seven 5-cycles. For most orders, $v \leq 53$, this is a perfect invariant, distinguishing all non-isomorphic graphs in $\mathcal{F}_v^2 \cup \mathcal{F}_v^3$.

We provide analytic proofs for $f_3(v)$ and $F_3(v), 7 \leq v \leq 10$, both to provide insight into the structure of such graphs, and also to illustrate the proof methods that are based on the radius 2 stars that motivated this research. Also provided are proofs of values for $f_r(v)$ where $f_r(v) < f(v) - 1$, and for $F_r(v)$ where $f_r(v) < f(v)$.

Non-bold values in the table were derived by simple examination of graphs known to be in $\mathcal{F}$. The non-bold values for $f_r(v)$ are equal to $f(v)$, and the sum of the non-bold values of $F_r(v)$ are the values of $F(v)$, with the exception of $v = 4$ for which there is a radius 1 graph in $\mathcal{F}_4$.

The bold values and bounds are for orders where $f_r(v) < f(v)$, and were derived by analytic and computational methods. For orders greater than 20, where $f_r(v) < f(v)$, we do not know if we have generated all of the non-isomorphic radius r extremal graphs, and therefore only state lower bounds for $F_r(V)$. Where the exact value of $f_r(v)$ is not known, the value of $F_r(v)$ is shaded to indicate that it is the number of graphs found with the size equal to the known lower bound on $f_r(v)$. In some of those cases we were able to narrow the bounds.

Case $f_3(7) = 7$ and $F_3(7) = 2$

Proof. The single graph $G \in \mathcal{F}_7$ has radius 2. Therefore $f_3(7) < 8$.

If there exists $G \in \mathcal{F}_7^3$, such that $e = 7$, G will contain a cycle. If $g = 5$ then G contains C_5, and the remaining two vertices will be pendant from vertices in the cycle, or they will form a P_2 pendant from a vertex in the cycle. In either case, such a graph will have radius 2.

If $g = 6$, then G is C_6 with an additional vertex pendant from a vertex in the cycle. This graph is 7a in Figure 6. If $g = 7$, G is C_7; this graph is 7b in Figure 6. □

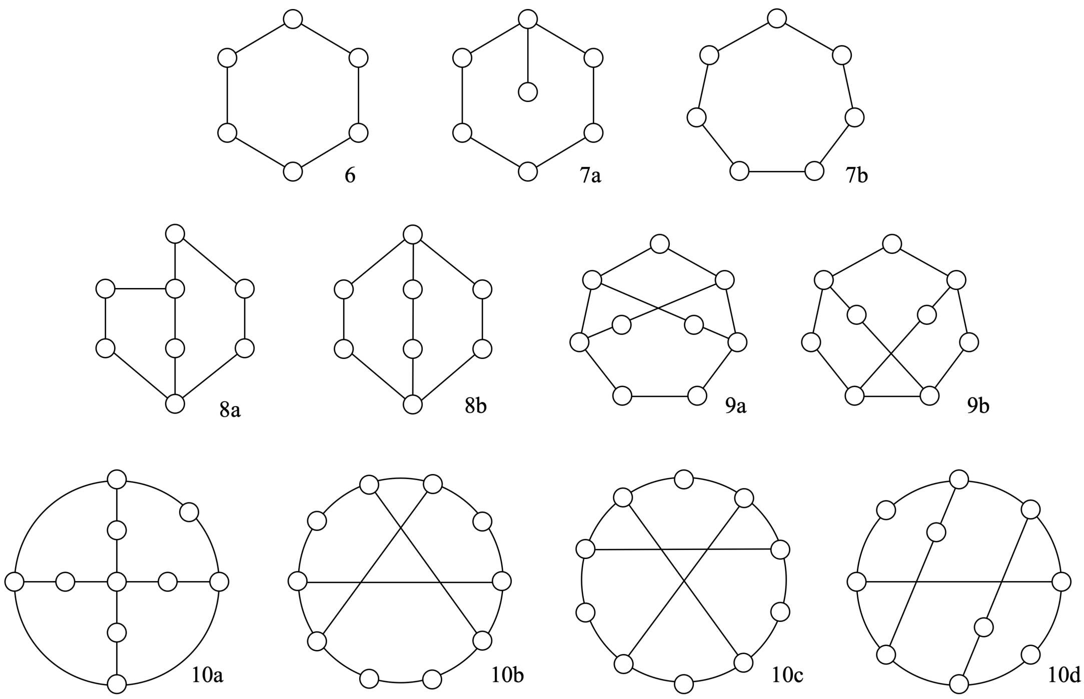

Figure 6: Radius 3 extremal graphs of orders 6 through 10

v	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$f_2(v)$	3	5	6	8	10	12	15	16	18	21	23	26	28	31	34
$f_3(v)$			6	**7**	**9**	**11**	**13**	16	18	**20**	23	**25**	28	31	34
$F_2(v)$	1	1	1	1	1	1	1	2	4	1	3	1	13	9	7
$F_3(v)$			1	**2**	**2**	**2**	**4**	1	3	**6**	1	**21**	9	5	8

v	19	20	21	22	23	24	25	26	27	28
$f_2(v)$	**37**	41	44	47	50	54	57	61	**64**	68
$f_3(v)$	38	**40**	**43**	**46**	**49**	**53**	57	**60**	65	68
$F_2(v)$	**10**	1	3	3	7	1	5	2	$\geq$ **9**	3
$F_3(v)$	1	**7**	$\geq$ **7**	$\geq$ **29**	$\geq$ **194**	$\geq$ **14**	1	$\geq$ **48**	1	1

v	29	30	31	32	33	34	35	36	37	38
$f_2(v)$	72	76	80	85	87	90	**94**	99	104	109
$f_3(v)$	**71**	**75**	**79**	**83-84**	87	90	95	99	104	109
$F_2(v)$	1	1	2	1	11	144	$\geq$ **187**	20	4	1
$F_3(v)$	$\geq$ **5**	$\geq$ **4**	$\geq$ **1**	≥ 1	1	93	5	16	3	1

v	39	40	41	42	43	44	45
$f_2(v)$	**113**	**118**	124	129	134	139	145
$f_3(v)$	114	120	**123**	**128**	**132-133**	**137-138**	**142-143**
$F_2(v)$	$\geq$ **22**	$\geq$ **12**	1	1	1	2	1
$F_3(v)$	1	1	$\geq$ **1**	$\geq$ **1**	≥ 2	≥ 1	≥ 2

v	46	47	48	49	50	51	52
$f_2(v)$	150	156	162	168	175	176	178
$f_3(v)$	**147-149**	**152-155**	**158-160**	**163-166**	**170-172**	176	178
$F_2(v)$	2	1	1	1	1	4	121
$F_3(v)$	≥ 2	≥ 7	≥ 1	≥ 16	≥ 1	3	27

v	53	54	55	56	57	58
$f_2(v)$	181	≥ 185	≥ 189	≥ 193	≥ 197	≥ 202
$f_3(v)$	181	$\geq$ **184**	$\geq$ **188**	$\geq$ **192**	≥ 197	$\geq$ **200**
$F_2(v)$	≥ 2647	≥ 13	≥ 5	≥ 11	≥ 3	≥ 1
$F_3(v)$	≥ 41	≥ 25	≥ 14	≥ 35	≥ 1	≥ 23

Table 1: Values and bounds for $f_2(v)$, $f_3(v)$, $F_2(v)$, and $F_3(v)$

Case $f_3(8) = 9$ **and** $F_3(8) = 2$

Proof. For the single graph in $\mathcal{F}_8$, $e = 10$ and $r = 2$, and thus $f_3(8) < 10$. If $f_3(8) = 9$, $G \in \mathcal{F}_8$ contains a cycle, and $g \geq 5$. The proof derives the elements of $\mathcal{F}_8$ by assuming girths 5 through 8. However, first we establish that $\delta(G) > 1$. That is, there can be no pendant vertices in G.

If $G \in \mathcal{F}_8^3$, and $\delta = 1$, then G is the unique graph G_7 in $\mathcal{F}_7$, shown in Figure 7, plus a pendant vertex adjacent to a leaf in the $(3, 1)$-star embedded in G_7. However, the center in G_7 labeled c would be at most distance 2 from any such pendant vertex, and $r(G) = 2$. Thus, $\delta(G) \neq 1$.

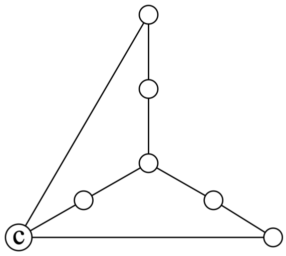

Figure 7: G_7, the unique graph in $\mathcal{F}_7$, with vertex c in its center

If $G \in \mathcal{F}_8^3$ and $g = 5$, then G has three vertices, x_1, x_2, and x_3, and four edges in addition to a C_5. If $x_i, 1 \leq i \leq 3$, is adjacent to two vertices in the C_5, then $g < 5$. Thus, each x_i is incident on at most one vertex in the C_5. Since G does not have any pendant vertices, the vertices x_i form P_3, with the end vertices of the P_3 adjacent to a distance 2 pair of vertices in the C_5. This is graph 8_a in Figure 6.

If $G \in \mathcal{F}_8^3$ and $g = 6$, then G has two vertices, x_1 and x_2, and three edges in addition to a C_6. If $x_i, 1 \leq i \leq 2$, is adjacent to two vertices in the C_6, then $g < 6$. Thus, each x_i is incident on at most one vertex in the C_6. Since G does not have any pendant vertices, the vertices x_i form P_2, with the end vertices of the P_2 adjacent to a distance 3 pair of vertices in the C_6. This is graph 8_b in Figure 6.

If $G \in \mathcal{F}_8^3$ and $g = 7$, then G has one vertex adjacent to a pair of vertices that are distance 3 apart in C_7, which implies that $g < 7$. And if $g = 8$, then G is C_8 plus an additional edge incident on a pair of vertices in the C_8, which implies $g < 8$. $\square$

Case $f_3(9) = 11$ **and** $F_3(9) = 2$

Proof. For the single graph $G \in \mathcal{F}_9$, $e = 12$ and $r = 2$. Thus, $f_3(9) < 12$. Graphs 9a and 9b in Figure 6 show that $f_3(9) = 11$.

If $G \in \mathcal{F}_9^3$, then by Propositions 2.6 and 2.7, $1 \leq \delta \leq 2$. If $\delta = 1$, then, since $f_3(9) = f(8) + 1$, $G = G_8$, the unique graph in $\mathcal{F}_8$, plus pendant vertex x. However, $r(G_8) = 2$, and all vertices in G_8 are either in the center, or adjacent to a center. Figure 8 shows G_8 with the vertices in the center shaded. Therefore, regardless of which vertex in G is adjacent to x, x will be at most distance 2 from some center

c in the G_8. And the eccentricity of c will also be 2 in G. Therefore, $\delta = 2$. Since $\delta = 2$, then by Propositions 2.6 and 2.7, $\Delta = 3$.

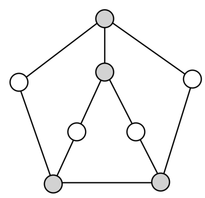

Figure 8: G_8, the unique graph in $\mathcal{F}_8$, with center vertices shaded

Given $\delta = 2$ and $\Delta = 3$, the degree sequence for G is $[2{:}5, 3{:}4]$, and there must be an adjacent pair of degree 3 vertices. Therefore, G contains star $S = S_{3[2,1,1]1}$. Since $v(S) = 8, e(S) = 7$, and $r = 3$, the remaining four edges must be incident only on leaves of S and x, external to S, and having either degree 2 or 3.

If $d(x) = 2$, x is either adjacent to the leaves in the 1-branches, or else to one leaf in a 1-branch and one leaf in the 2-branch. First assume that x is adjacent to a leaf in a 1-branch and to a leaf in the 2-branch. This is illustrated as graph A in Figure 9. The two remaining edges must be added from the set $\{(a, c), (a, d), (b, c), (c, d)\}$. There are three cases resulting from the six pairs of edges that can be added.

1. If $\{(a, c), (a, d)\}$, $\{(a, d), (b, c)\}$, or $\{(a, d), (c, d)\}$ is added, then $r(G) = 2$.
2. If $\{(a, c), (b, c)\}$ or $\{(b, c), (c, d)\}$ is added, then $g(G) = 4$.
3. If $\{(a, c), (c, d)\}$ is added, then $g(G) = 5$ and $r(G) = 3$. This is graph *9a* in Figure 6.

The edges that complete an element of $\mathcal{F}_9$ are shown as dashed lines.

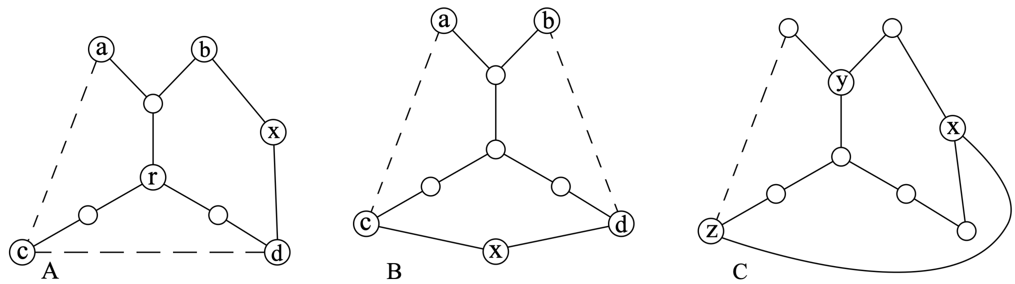

Figure 9: Construction of graphs in $\mathcal{F}_9^3$

Now, let x be adjacent to the leaves in the 1-branches. To satisfy the degree sequence, each of the the remaining two edges must be incident on a leaf in a 1-branch and a leaf in the 2-branch. The radius is 3. This is graph B in Figure 9, and *9b* in Figure 6. Graphs A and B in Figure 9 are nonisomorphic; A has an adjacent pair of degree 2 vertices, b and x, and B does not have such a pair.

If $d(x) = 3$, x must be adjacent to each of the leaves in the 1-branches in S, and to one of the leaves in the 2-branch in S. The remaining edge, shown as a dashed line in graph C in Figure 9, must be incident on the other leaf in the 2-branch and on either of the leaves in a 1-branch. Graph C is isomorphic to graph A in the figure. That can be seen by mapping (x, y, z) in C to (d, r, c) in A. □

Case $f_3(10) = 13$

Proof. The unique graph in $\mathcal{F}_{10}$ is the Petersen graph, with $e = 15$ and $r = 2$. Therefore, $f_3(10) < 15$.

Assume there exists G with $v = 10, e = 14, g \geq 5$, and $r = 3$. By Propositions 2.6 and 2.7, $1 \leq \delta \leq 2$ and $3 \leq \Delta \leq 4$. If $x \in V$ has degree 1, then, since $f(9) = 12$, G is the unique graph $G_9 \in \mathcal{F}_9$ with the addition of pendant vertex x. However, as Figure 10A shows, every vertex in G_9, which has radius 2, is either in the center of the graph, (the shaded vertices in the figure), or adjacent to a vertex in the center. Thus, if $\delta(G) = 1$, then $r(G) = 2$. Therefore, $\delta(G) = 2$.

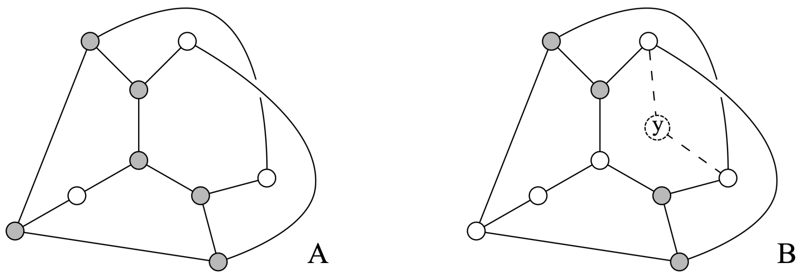

Figure 10: G_9, the unique graph in $\mathcal{F}_9$, with center vertices shaded and with a 10^{th} vertex, y

Now we consider the case where $\Delta = 4$. Since $r(G) = 3$, G must contain $S = S_{4,1}$ plus one vertex, x, external to S. Since $e(S) = 8$, then the induced subgraph on the four leaves of S plus x must have six edges. However, $f(5) = 5$. Therefore, $\Delta \neq 4$.

If $\Delta = 3$ then the degree sequence of G is $[2{:}2, 3{:}8]$. This implies that at least ten of the 14 edges in G are incident only on the eight degree 3 vertices in G. Since $f(8) = 10$, and there is a unique graph $G_8 \in \mathcal{F}_8$, G is G_8 plus two vertices, x and y, and four edges, each with at least one endpoint in $\{x, y\}$. Since G_8 has four degree 2 vertices, and all of those vertices have degree 3 in G, then $d(x) = d(y) = 2$, and x and y are not adjacent. Since $f(9) = f(8) + 2$, the subgraph in G induced by the G_8 + x is G_9, the unique graph in $\mathcal{F}_9$. Given the degree sequence for G, the remaining vertex, y, must be incident on two degree 2 vertices in the G_9 star, and those vertices must be leaves. Figure 10B shows y adjacent to those vertices. Though the resulting graph has girth 5, it has radius 2, as indicated by the shaded center vertices that are distance 2 from y. Thus, $\Delta(G) \neq 3$, and $f_3(10) < 14$. The graphs 10_a through 10_d show that $f_3(10) = 13$. □

Case $F_3(10) = 4$

Proof. We have established above, that if $G \in \mathcal{F}^3_{10}$, $\delta > 1$. By Proposition 2.7, $\delta < 3$. Therefore, $\delta = 2$. And, by Proposition 2.6, $3 \leq \Delta \leq 4$.

If $\Delta = 4$, and given $\delta = 2$, then G contains the star $S_{4,1,1}$ Let the leaves of S be $\{a, b, c, d\}$ and the external vertex be labeled x. Since $e(S) = 8$, then $e(\langle x, a, b, c, d\rangle) = 5$. The unique graph with $v = e = 5$ and $g \geq 5$ is C_5. The resulting graph is 10_a in Figure 6.

If $\Delta = 3$, then the degree sequence for G must be $[2{:}4, 3{:}6]$, and there must be a minimum of $13-(2\cdot4) = 5$ edges in the induced subgraph on the six vertices of degree 3 in G. Additionally, if a degree 3 vertex, x, were adjacent to three degree 3 vertices, x would be at the root of a $(3, 2)$-star, S. Since S would span G, then $r(G)$ would be 2. Thus, no degree 3 vertex is adjacent to three degree 3 vertices. Therefore, G contains $S = S_{3[2,2,1]1}$. Exhaustive search from S produced three non-isomorphic graphs with $\Delta = 3$. These are graphs $10_b, 10_c$, and 10_d in Figure 6. □

Case $F_3(13) = 6$

Proof. Since the unique graph in $\mathcal{F}_{13}$ has radius 2, $f_3(13) < 21$. From generated graphs, $f_3(13) = 20$. By Propositions 2.4 and 2.7, $2 \leq \delta \leq 3$.

If $\delta = 3$, then by Proposition 2.7, $\Delta \geq 4$. However, by Proposition 2.6, $\Delta < 4$. Therefore, $\delta = 2$. And since $f_3(13) = f(12) + 2$, we constructed the elements of $\mathcal{F}^3_{13}$ by adding a degree 2 vertex to each element of $\mathcal{F}_{12}$. That search yielded six non-isomorphic graphs with girth at least 5 and radius 3. □

Case $F_3(15) = 21$

Proof. Since the unique graph in $\mathcal{F}_{15}$ has radius 2, $f_3(15) < 26$. From generated graphs, $f_3(15) = 25$. By Propositions 2.4 and 2.7, $2 \leq \delta \leq 3$.

Since $f_3(15) = f(14) + 2$, we constructed all $G \in \mathcal{F}^3_{15}$ where $\delta(G) = 2$ by adding a degree 2 vertex, in all possible ways, to each element of $\mathcal{F}_{14}$ such that $g(G) \geq 5$ and $r(G) = 3$. This yielded nine non-isomorphic graphs in $\mathcal{F}^3_{15}$.

If $\delta = 3$, then by Propositions 2.6 and 2.7, $\Delta = 4$. Therefore, either $S_{4[2,2,2,2]2}$ or $S_{4[3,2,2,2]1}$ spans G. To reduce the search space for the edges amongst the leaves and external vertices, symmetries can be broken by fixing edges. In particular, since $\delta = 3$, there are four cases for fixing three edges incident on an external vertex x:

1. G contains $S_{4[2,2,2,2]2}$, and x is adjacent to one leaf in each of three branches;
2. G contains $S_{4[2,2,2,2]2}$, and x is adjacent to the other external vertex and to one leaf in each of two branches;
3. G contains $S_{4[3,2,2,2]1}$, and x is adjacent to a leaf in the 3-branch, and to a leaf in each of two 2-branches;

4. G contains $S_{4[3,2,2,2]1}$, and x is adjacent to a leaf in each 2-branch.

These four cases yielded 12 non-isomorphic graphs with $\delta = 3$ in $\mathcal{F}^3_{15}$. In addition to the nine graphs with $\delta = 2$, $F_3(15) = 21$. □

Case $F_2(19) = 10$

Proof. Since the Robertson graph is the unique graph in $\mathcal{F}_{19}$ [10], with 38 edges and radius 3, and we constructed radius 2 girth 5 graphs with $e = 37$, then $f_2(19) = 37$.

If $G \in \mathcal{F}^2_{19}$, then by Propositions 2.4 and 2.7, $\delta = 3$. Since $f_2(19) = f(18)+3 = 37$, we constructed all $G \in \mathcal{F}^2_{19}$ by exhaustively adding a degree 3 vertex to each graph in $\mathcal{F}_{18}$ such that $g(G) \geq 5$ and $r(G) = 2$. The search yielded ten non-isomorphic graphs in $\mathcal{F}^2_{19}$. □

Case $F_3(20) = 7$

Proof. Since the unique graph in $\mathcal{F}_{20}$ has radius 2, $f_3(20) < 41$. From generated graphs, $f_3(20) = 40$. By Propositions 2.4 and 2.7, for $G \in \mathcal{F}^3_{20}$, $2 \leq \delta \leq 4$.

If $\delta = 4$, then by Proposition 2.6, $\Delta = 4$. That is, G is 4-regular. The two non-isomorphic girth 5 4-regular graphs on 20 vertices were identified by Meringer [16].

Since $f_3(20) = f(18)+6$, if $\delta = 3$, and $V(G)$ has at least two vertices with degree 3, then G contains as a subgraph some graph in $\mathcal{F}_{18}$. And if $\delta(G) = 3$, and $V(G)$ has exactly one vertex x with degree 3, then since $\bar{d}(G) = 4$, the degree sequence for G is $[3{:}1, 4{:}18, 5{:}1]$. Thus, x must be adjacent to at least one vertex $y \in V(G)$, where $d(y) = 4$. Since $v(G - (x, y)) = 18$, and $e(G - (x, y)) = f_3(20) - 6 = f(18)$, then if $V(G)$ contains exactly one degree 3 vertex, G again contains as a subgraph some graph in $\mathcal{F}_{18}$.

If $\delta = 2$, then, since $f_3(20) = f(19) + 2$, G contains the unique graph $G_{19} \in \mathcal{F}_{19}$. However, since G_{19} is 4-regular, and $f(19) = f(18) + 4$, G_{19} contains as a subgraph some graph in $\mathcal{F}_{18}$. Therefore, if $V(G)$ contains a vertex of degree 2, G contains as a subgraph some graph in $\mathcal{F}_{18}$.

Every graph $G \in \mathcal{F}^3_{20}$, where $\delta(G) < 4$, contains an element of $\mathcal{F}_{18}$ as a subgraph, and therefore we constructed all such graphs by exhaustively adding edges to each graph in $\mathcal{F}_{18}$ augmented with two isolated vertices. The search yielded five non-isomorphic graphs. Together with the two graphs that are 4-regular, $F_3(20) = 7$. □

Case $f_2(40) = 118$

For $v \leq 53$, $f_2(40)$ is the unique case where $f_2(v) < f(v)-1$. In the other cases where $f_2(v) < f(v)$, we found by heuristic search graphs with $e = f(v) - 1$ and $r = 2$. For the case of $v = 40$ and $r = 2$, search only produced graphs with $e = f(40) - 2 = 118$. We prove that $f_2(40) = 118$.

Proof. The unique graph in $\mathcal{F}_{40}$ has $e = 120$ and $r = 3$. Therefore $f_2(40) < 120$, and heuristic search constructed graphs with $e = 118, g \geq 5$, and $r = 2$.

Since $f(39) = 114$, then if $G \in \mathcal{F}^2_{40}$ and $e = 119$, by Propositions 2.4 and 2.7, $\delta = 5$. If $x \in V(G)$ and $d(x) = 5$, then $G - x$ is the unique graph $G_{39} \in \mathcal{F}_{39}$. However, exhaustive search fails to add five edges to G_{39} plus an isolated vertex while maintaining girth 5 and radius 2. Therefore, $\delta \neq 5$ and $f_2(40) = 118$. □

Case $142 \leq f_3(45) \leq 143$

Proof. Since the unique graph in $\mathcal{F}_{45}$ has $e = 145$ and $r = 2$, and we constructed graphs with $v = 45, e = 142, g = 5$ and $r = 3$, then $142 \leq f_3(45) \leq 144$. We now lower the upper bound to 143.

If $f_3(45) = 144$, then for $G \in \mathcal{F}^3_{45}$, by Propositions 2.4 and 2.7, $5 \leq \delta(G) \leq 6$. If $\delta = 5$, and given $e(G) = f(44) + 5$, G can be generated by adding a degree 5 vertex, x, to either of the two graphs, G_{44a} and G_{44b}, in $\mathcal{F}_{44}$. However, a computer analysis shows that the radius of G_{44a} is 2, and that every vertex in it is either in the center, or is a neighbor of a center vertex. Thus, adding vertex x would construct a graph that also has radius 2. The same holds true for G_{44b}. Therefore, $\delta \neq 5$.

If $\delta = 6$, then by Propositions 2.6 and 2.7, $\Delta = 7$. The degree sequence for G is $[6{:}7, 7{:}38]$. Thus, even if all vertices of degree 6 are adjacent only to vertices of degree 7, the induced subgraph, H, on the degree 7 vertices has at least $144 - 6 \cdot 7 = 102$ edges. Since $\bar{d}(H) = (2 \cdot 102)/38 \approx 5.3, \Delta(H) \geq 6$. Therefore, there is at least one vertex, $x \in V(H)$ where $d(x) = 7$ in G, and $N(x)$ contains at least six vertices of degree 7 in G. Thus, G contains $S = S_{7[6,6,6,6,6,6,5]}$. But, $v(S) = 49 > v(G)$. Therefore, $\delta \neq 6$ and $f_3(45) \neq 144$. □

Case $158 \leq f_3(48) \leq 160$

Proof. Since the unique graph in $\mathcal{F}_{48}$ has $e = 162$ and $r = 2$, and we constructed a graph with $v = 48, e = 158, g = 5$, and $r = 3$, then $158 \leq f_3(48) \leq 161$. We lower the upper bound to 160.

If $f_3(48) = 161$, then by Propositions 2.4 and 2.7, if $G \in \mathcal{F}^3_{48}$, then $5 \leq \delta \leq 6$. Since $f(47) + 5 = 161$, if $\delta(G) = 5$, then G contains the unique subgraph $G_{47} \in \mathcal{F}_{47}$ plus additional vertex x adjacent to five vertices in the embedded subgraph. However, for all $y \in V(G_{47}), y \in \mathrm{cent}(G_{47}) \cup N(\mathrm{cent}(G_{47}))$. Therefore, x is at most distance 2 from some $z \in \mathrm{cent}(G_{47})$, and z has eccentricity of 2 in G. This implies that $r(G) = 2$, thus contradicting $\delta(G) = 5$.

If $\delta = 6$, then by Propositions 2.6 and 2.7, $\Delta = 7$. The degree sequence for G is $[6{:}14, 7{:}34]$. There must be at least 77 edges in the subgraph consisting of degree 7 vertices. That is, if $H = \langle \{x \in V(G) : d(x) = 7\} \rangle$, then $e(H) \geq (34 \cdot 7 - 14 \cdot 6)/2 = 77$. Therefore, $\bar{d}(H) \geq (77 \cdot 2)/34 \approx 4.5$. This implies there is at least one vertex $x \in V(G)$ where $d(x) = 7$ in G and x has at least five neighbors in G of degree 7. Therefore, with x as the root of star S that spans G, $S = S_{7[6,6,6,6,6,5,5]1}$. Since

$v(S) = 49 > v(G)$, $\delta(G) \neq 6$ and $f_3(48) \neq 161$. □

Case $163 \leq f_3(49) \leq 166$

Proof. Since the unique graph in $\mathcal{F}_{49}$ has $e = 168$ and $r = 2$, and we constructed graphs with $v = 49, e = 163, g = 5$, and $r = 3$, then $163 \leq f_3(50) \leq 167$. We lower the upper bound to 166.

If $f_3(49) = 167$, then for $G \in \mathcal{F}^3_{49}$, by Propositions 2.4 and 2.7, $5 \leq \delta(G) \leq 6$. If $\delta = 5$, and given $e(G) = f(48) + 5$, G can be generated by adding a degree 5 vertex, x, to the unique graph, G_{48}, in $\mathcal{F}_{48}$. However, for all $y \in V(G_{48}), y \in \mathrm{cent}(G) \cup N(\mathrm{cent}(G))$. Vertex x will be at most distance 2 from a center in G, and $r(G) = 2$. Therefore, $\delta \neq 5$.

If $\delta = 6$, then by Propositions 2.6 and 2.7, $\Delta = 7$. The degree sequence for G is $[6{:}9, 7{:}40]$. Thus, even if all vertices of degree 6 are adjacent only to vertices of degree 7, the induced subgraph, H, on the degree 7 vertices has at least $167 - 6 \cdot 9 = 113$ edges. Since $\bar{d}(H) = (2 \cdot 113)/40 = 5.65, \Delta(H) \geq 6$. Therefore, there is at least one vertex, $x \in V(H)$ where $d(x) = 7$ in G, and $N(x)$ in G contains at least six vertices of degree 7. Thus, G contains $S = S_{7[6,6,6,6,6,6,5]}$. However, $v(S) = 49$, and since $r(G) = 3$, G must have at least one additional vertex external to S. Therefore, $\delta \neq 6$ and $f_3(49) \neq 167$. □

Case $170 \leq f_3(50) \leq 172$

Proof. Since the Hoffman-Singleton graph, the Moore graph on 50 vertices, is the unique graph in $\mathcal{F}_{50}$, having $e = 175$ and $r = 2$, and we constructed a graph with $v = 50, e = 170, g = 5$, and $r = 3$, then $170 \leq f_3(50) \leq 174$. We lower the upper bound to 172. However, first we prove that $f_3(50) \neq 174$. This is a necessary step since it is possible that for order v there may not be a girth 5 radius 3 graph of size e, but there is one of size $e + 1$.

If $f_3(50) = 174$, then for $G \in \mathcal{F}^3_{50}$, by Propositions 2.4 and 2.7, $\delta(G) = 6$. Since $e(G) = f(49) + 6$, G can be generated by adding a degree 6 vertex, x, to the unique graph, G_{49}, in $\mathcal{F}_{49}$. However, for all $y \in V(G_{49}), y \in \mathrm{cent}(G) \cup N(\mathrm{cent}(G))$. Vertex x will be at most distance 2 from a center in G, and $r(G) = 2$. Therefore, $f_3(50) < 174$.

We now lower the upper bound to 172. If G has $r = 3$ and $e = 173$, then by Propositions 2.4 and 2.7, $5 \leq \delta \leq 6$.

If $\delta = 5$ then, since $173 = f(49) + 5$, G contains as a subgraph the unique graph $G_{49} \in \mathcal{F}_{49}$. However, for every $x \in V(G_{49}), x \in \mathrm{cent}(G_{49}) \cup N(\mathrm{cent}(G_{49}))$. Thus there is a center in G_{49} with eccentricity 2 in G. Therefore, $\delta(G) \neq 5$.

If $\delta = 6$, then by Propositions 2.6 and 2.7, $7 \leq \Delta \leq 8$. If $\Delta = 8$, then G contains $S = S_{8,5,1}$. Let $V(H)$ be the leaves of S plus the vertex external to S. Thus, $v(H) = 41$ and $e(H) = 173 - e(S) = 173 - 48 = 125$. However, $e(H) > f(41) = 124$. Therefore, $\Delta(G) \neq 8$.

If $\delta = 6$ and $\Delta = 7$, then the degree sequence for G is $[6{:}4, 7{:}46]$. Therefore, there must be at least $173 - 4 \cdot 6 = 149$ edges in the induced subgraph H on the set of 46 degree 7 vertices in G, and $\bar{d}(H) = 2 \cdot 149/46 \approx 6.5$. Therefore, $\Delta(H) \geq 7$, and at least one degree 7 vertex in G has seven degree 7 neighbors. Therefore G contains $S = S_{7,6}$. But since $v(S) = 50$, S spans G, and $r(G) = 2$. Therefore, $\delta(G) \neq 6$ and $f_3(50) \neq 173$. □

4 Conclusion

Because of the important role radius plays in the structure and analysis of girth 5 extremal graphs, we proved results on graphs constrained to radius 2 or to radius 3. We determined that, for $G \in \mathcal{F} \cup \mathcal{F}^2$ and $v \geq 5$,

1. the diameter of G is 2 or 3;
2. strengthening a result from [11], every $x \in V(G)$ is in a C_5 in G with the exception of degree 1 and 2 vertices when $v \in \{6, 11, 51,$ and possibly $3251\}$. For those cases, if $d(x) = 2$, x is in a C_5 or C_6 in G.

For $G \in \mathcal{F}^3 - \mathcal{F}$,

1. the diameter of G is 3 or 4;
2. every $x \in V(G)$, where $d(x) \geq 4$, is in a 5-cycle in G;
3. for $v \geq 9$, the girth of G is 5.

Since graph 7a in Figure 6 is the only known graph in $\mathcal{F}^3$ with a pendant vertex, we conjecture the following.

Conjecture 4.1 *For $v \geq 8$, if $G \in \mathcal{F}^3_v$, then $\delta(G) \geq 2$.*

Also, since graphs 7a and 8b in Figure 6 are the only known graphs in $\mathcal{F}^3$ with a degree 3 vertex not in a C_5, we state another conjecture.

Conjecture 4.2 *For $v \geq 9$, if $G \in \mathcal{F}^3_v$ and $d(x) = 3$, then x is in a C_5 in G.*

We proved that $f_3(v) > f_2(v-1)$. Though there are orders for which $f_2(v) > f_3(v)$ and for which $f_2(v) < f_3(v)$, we proved that

1. $f_r(v) < f_2(v)$ *if* $r \geq 4$;
2. $f_4(v) \leq f_3(v)$;
3. $f_r(v) < f_3(v)$ *if* $r \geq 5$.

Since we have not found graphs with radius 4 that prove $f_4(v) = f_3(v)$ for any order, we offer the following conjecture.

Conjecture 4.3 *For $v \geq 8, f_4(v) < f_3(v)$.*

Using analytic and computational methods, we determined exact values or narrow bounds for $f_2(v)$ and $f_3(v)$ for $v \leq 53$. Where $f_2(v)$ and $f_3(v)$ are known, we determined exact values or constructive lower bounds on $F_2(v)$ and $F_3(v)$. The graphs are available for download [9].

In addition to determining $f_2(v)$, $f_3(v)$, $\mathcal{F}_v^2$, and $\mathcal{F}_v^3$ for more orders v, a future direction for this work is to study the radius constrained extremal graphs for girths greater than 5.

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