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On restricted r-Stirling numbers, also known as r-Bessel numbers

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Abstract

The restricted r-Stirling number $\left\{ {n \atop k} \right\}_r^{\le 2}$, also known as an r-Bessel number, counts the partitions of an $(n+r)$-element set into $k+r$ blocks, where r distinguished elements have to belong to distinct blocks with the restriction that each block contains at most two elements ($0 \le k \le n$, $r \ge 0$). In this paper, we give a combinatorial investigation of these numbers and derive new identities. We also prove their log-concavity and unimodality properties through the study of restricted r-Bell polynomials.

1 Introduction

Stirling numbers of the first kind $\left[{n \atop k} \right]$ and Stirling numbers of the second kind $\left\{ {n \atop k} \right\}$ ($0 \le k \le n$) first appeared in James Stirling's Methodus Differentialis [23] as the coefficients in the polynomial expressions

$$x^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k} \right] x^k, \quad x^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\} x^{\underline{k}} \quad (n \ge 0)$$

between rising, falling factorials and ordinary powers, where the nth rising and falling factorial of x are defined by the products

$$x^{\overline{n}} = \prod_{j=0}^{n-1} (x+j), \quad x^{\underline{n}} = \prod_{j=0}^{n-1} (x-j) \quad (n \ge 0),$$

respectively. These numbers have well-known combinatorial meanings, namely, $\left[{n \atop k} \right]$ counts the permutations of the elements $1, \ldots, n$ which are the product of k disjoint

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cycles, while $\left\{ {n \atop k} \right\}$ is the number of partitions of the elements $1, \ldots, n$ into k nonempty subsets called blocks. If $n = 0$, as a degenerate case, we have only the empty permutation or partition with 0 cycles or blocks.

Carlitz [2], Broder [1] and Merris [16] independently introduced the widely studied r-generalization of Stirling numbers. For $0 \leq k \leq n$ and $r \geq 0$, a permutation or partition of the elements $1, \ldots, n + r$ is called an r-permutation or r-partition if $n + 1, \ldots, n + r$ belong to distinct cycles or blocks. These last r elements will be referred to as distinguished elements, and the cycles or blocks containing them as distinguished cycles or distinguished blocks. Then the r-Stirling number of the first kind $\left[{n \atop k} \right]_r$ counts the r-permutations of the elements $1, \ldots, n + r$ which have $k + r$ disjoint cycles in their cycle decomposition, while the r-Stirling number of the second kind $\left\{ {n \atop k} \right\}_r$ is the number of r-partitions of the elements $1, \ldots, n+r$ into $k+r$ blocks. (In other words, we have r distinguished and k non-distinguished cycles or blocks.) The r-Stirling numbers of both kinds also have equivalent characterizations by the polynomial identities

$$(x + r)^{\overline{n}} = \sum_{k=0}^{n} \left[{n \atop k} \right]_r x^k, \quad (x + r)^n = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r x^{\underline{k}} \quad (n \geq 0).$$

For further aspects of r-Stirling numbers, see [7, 11, 19, 20].

Bessel polynomials were introduced by Krall and Frink [13] as the polynomial solutions of certain second-order differential equations (see also [6]). Choi and Smith [4] found a combinatorial meaning of the reparametrized coefficients of Bessel polynomials and named them Bessel numbers. Since this interpretation has a close connection with Stirling numbers, Bessel numbers are also called restricted Stirling numbers.

The restricted Stirling number of the second kind $\left\{ {n \atop k} \right\}^{\leq 2}$ gives the number of partitions of the elements $1, \ldots, n$ into k blocks such that the cardinality of each block is at most 2 ($0 \leq k \leq n$). We note that partitioning elements into blocks containing at most two elements is essentially the same as arranging them into disjoint cycles of length at most 2. For this reason, we do not define restricted Stirling numbers of the first kind separately and simply omit the term "of the second kind" in the rest of the paper.

Cheon, Jung and Shapiro [3] combined the combinatorial definitions of the above two variants of Stirling numbers. Namely, the restricted r-Stirling number or r-Bessel number $\left\{ {n \atop k} \right\}_r^{\leq 2}$ counts the r-partitions of the elements $1, \ldots, n + r$ into $k + r$ blocks with the restriction that each block contains at most two elements ($0 \leq k \leq n$, $r \geq 0$). It is clear that these numbers give back restricted Stirling numbers if $r = 0$.

In Section 2, we contribute to the topic of restricted r-Stirling numbers with recurrences, polynomial identities, an explicit formula and some connections with r-Stirling and the so-called r-Whitney numbers. In Section 3, we study the real-rootedness of restricted r-Bell polynomials, which implies log-concavity and unimodality properties of the sequence of restricted r-Stirling numbers with a fixed upper parameter. Our proofs are mainly based on purely combinatorial ideas.

2 Restricted r-Stirling numbers

It follows from the definition of restricted r-Stirling numbers that $\left\{ {n \atop k} \right\}_r^{\leq 2} = 0$ if $0 \leq k \leq n$, $r \geq 0$ and $n > 2k + r$. Otherwise, in case of $n \leq 2k + r$, there exists at least one required r-partition, which contains $2k + r - n$ one-element and $n - k$ two-element blocks.

It is also easy to obtain the special values $\left\{ {n \atop 0} \right\}_r^{\leq 2} = r^{\underline{n}}$ and $\left\{ {n \atop n} \right\}_r^{\leq 2} = 1$ $(n, r \geq 0)$.

Various recurrences were derived for restricted r-Stirling numbers by combinatorial arguments in [10, 12], but in those expressions restricted $(r-1)$-Stirling numbers appear as well. Now, we present a recurrence relation which contains only restricted r-Stirling numbers. (We mention that this can be found in [9] for $r = 0$.)

Theorem 2.1. *If* $1 \leq k \leq n$ *and* $r \geq 0$*, then*

$$\left\{ {n+1 \atop k} \right\}_r^{\leq 2} = \left\{ {n \atop k-1} \right\}_r^{\leq 2} + (2k + r - n) \left\{ {n \atop k} \right\}_r^{\leq 2}.$$

Proof. We enumerate the r-partitions of the elements $1, \ldots, n + 1 + r$ into $k + r$ blocks, where each block contains one or two elements.

If the element $n + 1$ stands alone in a singleton, then we obviously have $\left\{ {n \atop k-1} \right\}_r^{\leq 2}$ possibilities. If it is in a two-element block, then the other $n + r$ elements can be r-partitioned into $k + r$ blocks containing at most two elements in $\left\{ {n \atop k} \right\}_r^{\leq 2}$ ways. As we have seen, this restricted r-Stirling number is equal to 0 if $n > 2k + r$. Otherwise, in case of $n \leq 2k + r$, the element $n + 1$ can be inserted into any of the $2k + r - n$ one-element blocks. $\square$

We can also prove a vertical recurrence for restricted r-Stirling numbers.

Theorem 2.2. *If* $0 \leq k \leq n$ *and* $r \geq 0$*, then*

$$\left\{ {n+1 \atop k+1} \right\}_r^{\leq 2} = \sum_{j=k}^{n} (2k + r + 1 - j)^{\underline{n-j}} \left\{ {j \atop k} \right\}_r^{\leq 2}.$$

Proof. We may suppose that $n \leq 2k + r + 1$, because otherwise both sides of the equality are equal to 0. We are interested in the number of r-partitions of the elements $1, \ldots, n + 1 + r$ into $k + 1 + r$ blocks, where each block contains at most two elements.

Denote by $n + 1 - j$ the smallest number among the maxima of the blocks ($j = k, \ldots, n$). First, we put this element into a block. Then the j non-distinguished and r distinguished elements greater than $n + 1 - j$ are all in the other $k + r$ blocks, and each of these blocks contains at least one of them. Therefore, they can be

r-partitioned into $k+r$ blocks having cardinality at most 2 in $\left\{ {j \atop k} \right\}_r^{\le 2}$ ways. (We note that $\left\{ {j \atop k} \right\}_r^{\le 2} = 0$ if $j = n = 2k+r+1$.) At present, we have $2k+r+1-j$ one-element blocks, hence the $n-j$ elements smaller than $n+1-j$ can be placed into them in $(2k+r+1-j)^{\underline{n-j}}$ ways. $\square$

To state the next theorem, we need to define generalized rising and falling factorials with difference m as

$$(x|m)^{\overline{n}} = \prod_{j=0}^{n-1}(x+mj), \quad (x|m)^{\underline{n}} = \prod_{j=0}^{n-1}(x-mj) \quad (m \ge 1, n \ge 0).$$

The first of the following polynomial identities shows that restricted r-Stirling numbers are the transition coefficients between shifted ordinary falling factorials and falling factorials with difference 2.

Theorem 2.3. *If $n, r \ge 0$, then*

$$(x+r)^{\underline{n}} = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{\le 2} (x|2)^{\underline{k}},$$

$$(x-r)^{\overline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\}_r^{\le 2} (x|2)^{\overline{k}}.$$

Proof. The second equality follows from the first one by substituting $-x$.

To prove the first identity, we enumerate the r-partitions of the elements $1, \ldots, n+r$ into blocks containing one or two elements, where the largest element of each non-distinguished block is coloured twice: the primary colour is chosen from two colours, while the secondary colour is chosen from c colours ($c \ge n$) so that the secondary colours of distinct elements are different.

If we have k non-distinguished blocks ($k = 0, \ldots, n$), then the number of these coloured r-partitions is clearly $\left\{ {n \atop k} \right\}_r^{\le 2} 2^k c^{\underline{k}} = \left\{ {n \atop k} \right\}_r^{\le 2} (2c|2)^{\underline{k}}$, hence their total number is $\sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{\le 2} (2c|2)^{\underline{k}}$.

Alternatively, we put the distinguished elements into distinct blocks first, then we place the non-distinguished elements in decreasing order and colour them if necessary. Suppose that j non-distinguished elements are already done ($j = 0, \ldots, n-1$). Denote by t and $2s$ the number of elements among them which were placed into non-distinguished blocks and two-element non-distinguished blocks, respectively. It means that $t-2s$ elements stand in non-distinguished singletons, and $j-t$ non-distinguished elements are in distinguished blocks up to this point.

Then the next element $n-j$ can be placed into any of the other $r-(j-t)$ distinguished blocks, into one of the non-distinguished singletons, or it can open a new block. Because of the decreasing order, this element has to be coloured only in the latter case, but $t-s$ of the c secondary colours were previously used. Consequently, we have $r-(j-t)+t-2s+2(c-(t-s)) = 2c+r-j$ possibilities to place and colour the element $n-j$, hence the total number of coloured r-partitions is $\prod_{j=0}^{n-1}(2c+r-j) = (2c+r)^{\underline{n}}$. □

A simple combinatorial argument gives the explicit formula

$$\left\{ n \atop k \right\}_r^{\leq 2} = \sum_{j=0}^{\min\{n-k,r\}} \binom{r}{j} n^{\underline{2n-2k-j}} \frac{1}{2^{n-k-j}(n-k-j)!}$$

for restricted r-Stirling numbers ($0 \leq k \leq n$, $r \geq 0$), this is equivalent to the expression in [3].

Yang and Qiao [24] found another explicit formula for restricted Stirling numbers which is similar to the usual one for classical Stirling numbers of the second kind. We generalize it and offer a different proof by using the inclusion–exclusion principle.

Theorem 2.4. *If $0 \leq k \leq n$ and $r \geq 0$, then*

$$\left\{ n \atop k \right\}_r^{\leq 2} = \frac{1}{2^k k!} \sum_{j=0}^{k} (-1)^j \binom{k}{j} (2(k-j)+r)^{\underline{n}}.$$

Proof. Let $A = \{1, \ldots, n+r\}$ with distinguished elements $n+1, \ldots, n+r$ and B be a $(k+r)$-element set. We consider the coloured surjective functions $A \to B$ with the following properties:

- every element of B is the image of at most two elements of A,
- distinguished elements of A have different images,
- every element of B which is not the image of a distinguished element is coloured with one of two colours.

The collection of the preimages of the elements of B forms an r-partition of the elements of A into $k+r$ blocks containing at most two elements. After the assignment of these preimages to the elements of B and the colouring, we find that the number of our coloured surjective functions is $2^k(k+r)!\left\{ n \atop k \right\}_r^{\leq 2}$.

Let X be the set of coloured functions $A \to B$ with the above three properties, and denote by Y_i the set of coloured functions in X for which the ith element of B does not appear as an image ($i = 1, \ldots, k+r$).

To determine the cardinality of X, we begin with choosing the images of the distinguished elements, which can be done in $(k+r)^{\underline{r}}$ ways. Suppose that the first l non-distinguished elements already have their images ($l = 0, \ldots, n-1$). Denote by t the number of elements among them whose images differ from the images of the distinguished elements, and assume that there are $2s$ elements among these t elements which have pairwise the same image. Then the image of the next element $l+1$ can be an element of B which is

- the image of a distinguished element, but it is not the image of a previous non-distinguished element. The number of such elements in B is $r-(l-t)$.
- the image of exactly one previous non-distinguished element, but it is not the image of a distinguished element. The number of such elements in B is $t-2s$, and we note that they are already coloured.
- not the image of any element yet. The number of such elements in B is $k-(t-s)$, and the chosen element has to be coloured with one of the two colours.

These yield $r-(l-t)+t-2s+2(k-(t-s)) = 2k+r-l$ possibilities altogether. Therefore, $|X| = (k+r)^{\underline{r}}(2k+r)^{\underline{n}}$.

Since the r distinguished elements have different images, the intersection of more than k sets of type Y_i is empty. Or else if $1 \le j \le k$, then similar arguments give that the cardinality of the intersection of j sets of type Y_i is $(k+r-j)^{\underline{r}}(2(k-j)+r)^{\underline{n}}$. Then, by the inclusion–exclusion principle, the number of coloured surjective functions with the above three properties is

$$|X \setminus (Y_1 \cup \cdots \cup Y_{k+r})| = \sum_{j=0}^{k} (-1)^j \binom{k+r}{j} (k+r-j)^{\underline{r}} (2(k-j)+r)^{\underline{n}},$$

from which the formula stated in the theorem follows after some simplification. □

Recently, Stenlund [22] studied polynomials whose coefficients are products of Stirling numbers of the first and second kind. It turns out that a special evaluation of these polynomials gives signed restricted Stirling numbers, this fact already appeared in [15, 24]. We generalize this result and present a proof simply based on polynomial identities of the combinatorial numbers in question.

For $0 \le k \le n$ and $r, s \ge 0$, introduce the polynomial

$$p_{n,k,r,s}(x) = \sum_{j=k}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_r \begin{Bmatrix} j \\ k \end{Bmatrix}_s x^{j-k}.$$

Clearly, we have

$$p_{n,k,r,s}(0) = \begin{bmatrix} n \\ k \end{bmatrix}_r .$$

Moreover, theorems in [1, 2] and [19] state that

$$p_{n,k,r,s}(-1) = \binom{n}{k} (r-s)^{\overline{n-k}},$$

and

$$p_{n,k,r,s}(1) = \left\lfloor \begin{matrix} n \\ k \end{matrix} \right\rfloor_{\frac{r+s}{2}}$$

is an $\frac{r+s}{2}$-Lah number if r, s have the same parity.

We show that the result of substituting -2 in place of the indeterminate of a polynomial of type $p_{n,k,r,s}(x)$ is a signed restricted $(2s-r)$-Stirling number.

Theorem 2.5. *If* $0 \le k \le n$ *and* $0 \le r \le 2s$, *then*

$$p_{n,k,r,s}(-2) = (-1)^{n-k} \left\{ {n \atop k} \right\}_{2s-r}^{\le 2}.$$

Proof. On the one hand, Theorem 2.3 gives

$$(x-(2s-r))^{\overline{n}} = \sum_{k=0}^{n} (-1)^{n-k} \left\{ {n \atop k} \right\}_{2s-r}^{\le 2} (x|2)^{\overline{k}}.$$

On the other hand, we apply the polynomial identities of r-Stirling numbers of the first kind and s-Stirling numbers of the second kind to have

$$\begin{aligned} ((x-2s)+r)^{\overline{n}} &= \sum_{j=0}^{n} \left[{n \atop j} \right]_r (x-2s)^j = \sum_{j=0}^{n} \left[{n \atop j} \right]_r 2^j \left(\frac{x}{2} - s \right)^j \\ &= \sum_{j=0}^{n} \left[{n \atop j} \right]_r 2^j \sum_{k=0}^{j} (-1)^{j-k} \left\{ {j \atop k} \right\}_s \left(\frac{x}{2} \right)^{\overline{k}} \\ &= \sum_{k=0}^{n} \sum_{j=k}^{n} \left[{n \atop j} \right]_r \left\{ {j \atop k} \right\}_s (-2)^{j-k} (x|2)^{\overline{k}}. \end{aligned}$$

By comparing these two expressions, we obtain

$$\sum_{j=k}^{n} \left[{n \atop j} \right]_r \left\{ {j \atop k} \right\}_s (-2)^{j-k} = (-1)^{n-k} \left\{ {n \atop k} \right\}_{2s-r}^{\le 2}.$$

□

The appearance of falling and rising factorials with difference 2 in Theorem 2.3 suggests some possible connection between restricted r-Stirling numbers and r-Whitney numbers. We only need r-Whitney numbers of the second kind which were introduced by Mező [17] through polynomial identities, however, they can be found earlier in [5] under a different name. We recall the combinatorial interpretation of these numbers from [8] with a small modification: An r-partition is called a Whitney coloured r-partition with m colours if

- the largest elements of the blocks are uncoloured,
- elements in distinguished blocks are uncoloured,
- each remaining element is coloured with one of the m colours.

Then, for $0 \le k \le n$, $m \ge 1$ and $r \ge 0$, the r-Whitney number of the second kind $W_{m,r}(n,k)$ counts the Whitney coloured r-partitions of the elements $1, \ldots, n+r$ into $k+r$ blocks with m colours.

Jung, Mező and Ramírez [10] found that certain combinations of Stirling or r-Stirling numbers of the second kind with restricted r-Stirling numbers give r-Whitney or $(2r)$-Whitney numbers of the second kind with two colours, respectively. In the next theorem, we generalize this result and prove it in a purely combinatorial manner.

Theorem 2.6. *If $0 \leq k \leq n$ and $r, s \geq 0$, then*

$$\sum_{j=k}^{n} \left\{ {n \atop j} \right\}_r \left\{ {j \atop k} \right\}_s^{\leq 2} = W_{2,r+s}(n,k).$$

Proof. The number of Whitney coloured $(r+s)$-partitions of the elements $1, \ldots, n+r+s$ into $k+r+s$ blocks with two colours such that the largest elements of the non-distinguished blocks are additionally coloured with one of the same two colours is $2^k W_{2,r+s}(n,k)$.

Considering such an extended Whitney coloured $(r+s)$-partition with two colours, we can construct subblocks as follows:

- From a distinguished block containing one of the elements $n+r+1, \ldots, n+r+s$, we obtain a subblock by deleting the distinguished element (if the result of the deletion is the empty set, then it is not handled as a subblock).
- A distinguished block containing one of the elements $n+1, \ldots, n+r$ is a subblock.
- A non-distinguished block whose elements are of the same colour is a subblock.
- A non-distinguished block which contains elements of both colours is split into two subblocks according to the colours.

It is clear that the collection of subblocks forms an r-partition of the elements $1, \ldots, n+r$, the number of subblocks is at least $k+r$ and cannot exceed $n+r$.

Now, we count the extended Whitney coloured $(r+s)$-partitions with two colours described at the beginning of the proof through the subblocks. First, we r-partition the elements $1, \ldots, n+r$ into $j+r$ subblocks ($j = k, \ldots, n$). The subblocks containing one of the elements $n+1, \ldots, n+r$ are original distinguished blocks. Thereafter, we s-partition the other j subblocks and the distinguished elements $n+r+1, \ldots, n+r+s$ into $k+s$ blocks such that a subblock is allowed to share its block with at most one other subblock or a distinguished element.

- A block which contains only a single distinguished element or consists of a distinguished element and a subblock gives a distinguished block in the original sense.
- If a block contains only one subblock, then this subblock is an original block, and its elements are uniformly coloured with one of the two colours.

- If a block consists of two subblocks, then their union is an original block, while we can decide about the colours of the elements by subblocks in two ways.

Therefore, the number of the required extended Whitney coloured $(r+s)$-partitions with two colours is $2^k \sum\limits_{j=k}^{n} \left\{ {n \atop j} \right\}_r \left\{ {j \atop k} \right\}_s^{\le 2}$. $\square$

3 Restricted r-Bell polynomials

The coefficients of the well-known Bell polynomials are Stirling numbers of the second kind with a fixed upper parameter. Similarly, we can define the nth restricted r-Bell polynomial as

$$B_{n,r}^{\le 2}(x) = \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{\le 2} x^k$$

for $n, r \ge 0$. These polynomials were introduced by Jung, Mező and Ramírez [10], but they already appeared much earlier in a paper of Miksa, Moser and Wyman [18] in case of $r = 0$.

The recurrence of restricted r-Stirling numbers implies the following first-order recurrence relation for restricted r-Bell polynomials.

Theorem 3.1. *If $n, r \ge 0$, then*

$$B_{n+1,r}^{\le 2}(x) = (x + r - n) B_{n,r}^{\le 2}(x) + 2x (B_{n,r}^{\le 2}(x))'.$$

Proof. The statement can be easily verified for $n = 0$. If $n \ge 1$, then Theorem 2.1 gives

$$\begin{aligned}
B_{n+1,r}^{\le 2}(x) &= \sum_{k=0}^{n+1} \left\{ {n+1 \atop k} \right\}_r^{\le 2} x^k = \sum_{k=1}^{n} \left\{ {n+1 \atop k} \right\}_r^{\le 2} x^k + r^{\underline{n+1}} + x^{n+1} \\
&= \sum_{k=1}^{n} \left\{ {n \atop k-1} \right\}_r^{\le 2} x^k + \sum_{k=1}^{n} (2k + r - n) \left\{ {n \atop k} \right\}_r^{\le 2} x^k + r^{\underline{n+1}} + x^{n+1} \\
&= \sum_{k=0}^{n-1} \left\{ {n \atop k} \right\}_r^{\le 2} x^{k+1} + 2 \sum_{k=1}^{n} k \left\{ {n \atop k} \right\}_r^{\le 2} x^k + (r-n) \sum_{k=1}^{n} \left\{ {n \atop k} \right\}_r^{\le 2} x^k \\
&\quad + r^{\underline{n+1}} + x^{n+1} \\
&= x \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{\le 2} x^k + 2x \sum_{k=1}^{n} k \left\{ {n \atop k} \right\}_r^{\le 2} x^{k-1} + (r-n) \sum_{k=0}^{n} \left\{ {n \atop k} \right\}_r^{\le 2} x^k \\
&= (x + r - n) B_{n,r}^{\le 2}(x) + 2x (B_{n,r}^{\le 2}(x))'.
\end{aligned}$$

$\square$

Jung, Mező and Ramírez [10] studied the roots of restricted r-Bell polynomials. They proved that all roots of the polynomial $B_{n,0}^{\le 2}(x)$ are non-positive real numbers ($n \ge 1$). They also formulated the conjecture that the same holds in general, for arbitrary $r \ge 0$, which we confirm in the next theorem by using graph theory.

Theorem 3.2. *If $n \geq 1$ and $r \geq 0$, then all roots of the polynomial $B_{n,r}^{\leq 2}(x)$ are real and non-positive.*

Proof. It is easy to observe that the definition of restricted r-Stirling numbers can be translated into the language of graph theory (see [3], also [9] for $r = 0$). Consider a complete $(n+1)$-partite graph which has n one-element and one r-element partite sets. In this graph, the number of $(n-k)$-element matchings is equal to $\left\{ {n \atop k} \right\}_r^{\leq 2}$ if $0 \leq k \leq n$.

It is known that all roots of the matching generating polynomial of a loopless graph are real and negative (see, e.g., [14]). Since the reciprocal polynomial of $B_{n,r}^{\leq 2}(x)$ is the matching generating polynomial of the above complete multipartite graph, the roots of $B_{n,r}^{\leq 2}(x)$ are the reciprocals of the roots of the matching generating polynomial possibly together with 0, consequently they are all real and non-positive. $\square$

Remark. We note that the approach to restricted r-Stirling numbers in the proof shows an interesting similarity to the graph theoretic interpretation of r-Lah numbers in [20].

Log-concavity and unimodality of the sequence of restricted r-Stirling numbers with a fixed upper parameter are immediate consequences of the previous result together with Newton's theorem (see, e.g., [21]). For $r = 0$, these properties can already be found in [4, 9, 10].

Corollary 3.3. *If $n \geq 1$ and $r \geq 0$, then the sequence $\left(\left\{ {n \atop k} \right\}_r^{\leq 2} \right)_{k=0}^{n}$ is log-concave and unimodal.*

We can easily derive the multiplicity of 0 as a root of restricted r-Bell polynomials from the basic properties of restricted r-Stirling numbers. We conjecture a stronger assertion about the negative roots, namely we expect that they are all simple.

Conjecture 3.4. *Let $n \geq 1$ and $r \geq 0$. If $n \leq r$, then all roots of the polynomial $B_{n,r}^{\leq 2}(x)$ are negative and simple. If $n > r$ and*

- *$n \equiv r \pmod 2$, then 0 has multiplicity $\frac{n-r}{2}$, the other roots of the polynomial $B_{n,r}^{\leq 2}(x)$ are negative and simple,*
- *$n \not\equiv r \pmod 2$, then 0 has multiplicity $\frac{n-r+1}{2}$, the other roots of the polynomial $B_{n,r}^{\leq 2}(x)$ are negative and simple.*

We prove a partial result concerning this conjecture.

Theorem 3.5. *Let $n \geq 1$, $r \geq 0$ and $n \equiv r \pmod 2$. If the assertion in the conjecture holds for the polynomial $B_{n,r}^{\leq 2}(x)$, then it also holds for the polynomial $B_{n+1,r}^{\leq 2}(x)$.*

Proof. The key observation we need is the identity

$$\left(e^{\frac{1}{2}x}x^{\frac{r-n}{2}}B_{n,r}^{\leq 2}(x)\right)' = \frac{1}{2}e^{\frac{1}{2}x}x^{\frac{r-n}{2}-1}B_{n+1,r}^{\leq 2}(x) \quad (x \in \mathbb{R}),$$

which follows from Theorem 3.1.

If $n < r$ and the polynomial $B_{n,r}^{\leq 2}(x)$ has n negative simple roots, then $e^{\frac{1}{2}x}x^{\frac{r-n}{2}}B_{n,r}^{\leq 2}(x)$ has $n+1$ zeros, one of them is 0, the others are negative. Furthermore, we have $\lim\limits_{x\to-\infty} e^{\frac{1}{2}x}x^{\frac{r-n}{2}}B_{n,r}^{\leq 2}(x) = 0$. Then Rolle's mean value theorem gives that $\left(e^{\frac{1}{2}x}x^{\frac{r-n}{2}}B_{n,r}^{\leq 2}(x)\right)'$ has $n+1$ negative zeros, which are the roots of the polynomial $B_{n+1,r}^{\leq 2}(x)$ in view of the above identity.

If $n = r$ and the polynomial $B_{n,r}^{\leq 2}(x)$ has n negative simple roots, then we can similarly show that the polynomial $B_{n+1,r}^{\leq 2}(x)$ has n distinct negative roots, while 0 is an additional simple root.

Finally, if $n > r$ and the polynomial $B_{n,r}^{\leq 2}(x)$ has $\frac{n+r}{2}$ negative simple roots beside 0 as a root of multiplicity $\frac{n-r}{2}$, then $x^{\frac{r-n}{2}}B_{n,r}^{\leq 2}(x)$ is a polynomial with the same $\frac{n+r}{2}$ negative simple roots. It follows again from Rolle's theorem that $\left(e^{\frac{1}{2}x}x^{\frac{r-n}{2}}B_{n,r}^{\leq 2}(x)\right)'$ has $\frac{n+r}{2}$ negative zeros. By the identity at the beginning of the proof, all of them are roots of the polynomial $B_{n+1,r}^{\leq 2}(x)$, which has 0 as an additional root of multiplicity $\frac{n-r}{2}+1$. □

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