

On cocliques in commutative Schurian association schemes of the symmetric group

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Abstract

Given the symmetric group $G = \text{Sym}(n)$ and a multiplicity-free subgroup $H \leq G$, the orbitals of the action of G on G/H by left multiplication induce a commutative association scheme. The irreducible constituents of the permutation character of G acting on G/H are indexed by partitions of n and if $\lambda \vdash n$ is the second largest partition in dominance ordering among these, then the Young subgroup $\text{Sym}(\lambda)$ admits two orbits in its action on G/H , which are \mathcal{S}_λ and its complement. In their 2016 monograph, Godsil and Meagher asked whether \mathcal{S}_λ is a coclique of a graph in the commutative association scheme arising from the action of G on G/H . If such a graph exists, they also asked whether its smallest eigenvalue is afforded by the λ -module.

In this paper, we initiate the study of this question by taking $\lambda = [n - 1, 1]$. We show that the answer to this question is affirmative for the pair of groups (G, H) , where $G = \text{Sym}(2k + 1)$ and $H = \text{Sym}(2) \wr \text{Sym}(k)$, or $G = \text{Sym}(n)$ and H is one of $\text{Alt}(k) \times \text{Sym}(n - k)$, $\text{Alt}(k) \times \text{Alt}(n - k)$, or $(\text{Alt}(k) \times \text{Alt}(n - k)) \cap \text{Alt}(n)$. For the pair $(G, H) = (\text{Sym}(2k), \text{Sym}(k) \wr \text{Sym}(2))$, we also prove that the answer to this question of Godsil and Meagher is negative.

1 Introduction

Let X be a finite non-empty set and let $\mathcal{R} = \{R_0, R_1, \dots, R_k\}$ be a collection of relations on X . We say that the pair (X, \mathcal{R}) is an *association scheme* if the following statements are satisfied:

- (i) $\{(x, x) : x \in X\} \in \mathcal{R}$,
- (ii) \mathcal{R} is a partition of $X \times X$,
- (iii) for any $i \in \{0, 1, \dots, k\}$, the relation $\{(y, x) : (x, y) \in R_i\}$ belongs to \mathcal{R} ,
- (iv) for any $(x, y) \in R_k$, the number $|\{z \in X : (x, z) \in R_i \text{ and } (z, y) \in R_j\}|$ is a number p_{ij}^k depending only on i, j and k and not the choice of x and y .

The size of X is the *order* of the association scheme (X, \mathcal{R}) and the number of relations, $k + 1$, is its *rank*. It is worth noting that a non-empty set X and a relation R on X determine a digraph whose vertex set is X and for any $x, y \in V(X)$, an arc between x and y occurs if and only if $(x, y) \in R$. Consequently, each relation in an association scheme determines a digraph on X and therefore an adjacency matrix. An association scheme (X, \mathcal{R}) is *symmetric* if the adjacency matrices corresponding to the relations are symmetric. Moreover, (X, \mathcal{R}) is *commutative* if the corresponding adjacency matrices commute with each other. A survey on commutative association schemes can be found in [16].

An example of well-known association schemes is the *triangular association scheme* (X, \mathcal{R}) , where $X = \{A \subset \{1, 2, \dots, n\} : |A| = 2\}$ and $\mathcal{R} = \{R_0, R_1, R_2\}$, with $R_i = \{(A, B) \in X \times X : |A \cap B| = 2 - i\}$, for $i \in \{0, 1, 2\}$.

Association schemes were introduced in the 1950s by Bose and Shimamoto [4], and the study of these objects has developed into a major area of study in algebraic combinatorics since then. A generalization of these objects known as *coherent configurations* were also introduced by Higman [13] in the 1970s to study permutation groups. Another well-known example of an association scheme arises from the action of a finite transitive group $G \leq \text{Sym}(\Omega)$, where Ω is a finite non-empty set. If \mathcal{O} is the set of all orbitals of the action of G on Ω , i.e., its orbits in the induced action on $\Omega \times \Omega$, then (Ω, \mathcal{O}) is an association scheme.

The association scheme (Ω, \mathcal{O}) is called the *orbital scheme* of G . In addition, an association scheme arising as an orbital scheme is called *schurian*. The (di)-graphs in an orbital scheme are called orbital (di)-graphs. An example of schurian association schemes is again the triangular association scheme which arises from the action of $\text{Sym}(n)$ on the 2-subsets of $\{1, 2, \dots, n\}$.

Next, we recall a result about the commutativity of schurian association schemes. Let H be a subgroup of a group G . We will denote the trivial character of H by $\mathbf{1}_H$ and the induced character of $\mathbf{1}_H$ on the group G by $\mathbf{1}_H^G$. Let $\text{Irr}(G) = \{\phi_1, \phi_2, \dots, \phi_t\}$ be the set of (complex) irreducible characters of G . We say that H is a *multiplicity-*

free subgroup of G if the irreducible decomposition

$$\mathbf{1}_H^G = \sum_{i=1}^t m_i \phi_i,$$

is such that $m_i \in \{0, 1\}$, for $i \in \{1, \dots, t\}$. If $H \leq G$ is multiplicity-free, then we say that the pair (G, H) is a *Gelfand pair*. Gelfand pairs are well studied and we refer the reader to [6, Chapter 4] for details on them. A consequence of (G, H) being a Gelfand pair is that the Bose-Mesner algebra of the schurian association scheme induced by the action of G on G/H coincides with its Hecke algebra (also known as the double centralizer algebra). The commutativity of schurian association schemes is determined by the point stabilizer of the corresponding transitive groups. Let $G \leq \text{Sym}(\Omega)$ be a transitive group and let H be the point stabilizer of G . The association scheme which arises from the transitive group G is commutative if and only if (G, H) is a Gelfand pair (see [11, Chapter 13] for details).

As we will study commutative schurian association schemes arising from transitive actions of the symmetric group, we need to review some facts on the representation theory of these groups. Recall that the irreducible submodules of $\mathbb{C} \text{Sym}(n)$ are indexed by partitions of the integer n . For any partition $\lambda \vdash n$, the irreducible $\mathbb{C} \text{Sym}(n)$ -module corresponding to λ is the *Specht module* S^λ . Therefore, the irreducible characters of $\text{Sym}(n)$ are also indexed by the partitions of the integer n . Given a partition $\lambda \vdash n$, we will denote the corresponding irreducible character by χ^λ . For any partition $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ of n , the *Young subgroup* $\text{Sym}(\lambda)$ is the subgroup $\text{Sym}(\lambda_1) \times \text{Sym}(\lambda_2) \times \dots \times \text{Sym}(\lambda_k)$. For any two partitions $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k]$ and $\mu = [\mu_1, \mu_2, \dots, \mu_t]$ of n , we say that λ *dominates* μ and write $\mu \trianglelefteq \lambda$, if $\sum_{i=1}^j \lambda_i \geq \sum_{i=1}^j \mu_i$, for all $j \in \{1, 2, \dots, t\}$.

Now, we will describe the problem considered in this paper. Henceforth, we assume that $G = \text{Sym}(n)$ and H is a multiplicity-free subgroup of G . Let

$$\Lambda(n, H) := \{ \lambda \vdash n : \langle \mathbf{1}_H^G, \chi^\lambda \rangle = 1 \}.$$

The eigenvalues of the orbital digraphs corresponding to (G, H) are indexed by the partitions in $\Lambda(n, H)$. In particular, an eigenspace of any orbital digraph is a direct sum of certain irreducible $\mathbb{C} \text{Sym}(n)$ -modules, whose corresponding partitions appear in $\Lambda(n, H)$. For any $\lambda \in \Lambda(n, H)$, we refer to the subspace given by the Specht module S^λ as the λ -module. The partition $[n]$ is always an element of $\Lambda(n, H)$ due to the fact that G acts transitively on G/H by left multiplication. Let $\lambda \neq [n]$ be the partition which is the second largest in dominance ordering in $\Lambda(n, H)$. By [11, Theorem 13.9.1], the Young subgroup $\text{Sym}(\lambda)$ admits two orbits in its action on G/H . In [11, Problem 16.13.1], Godsil and Meagher asked the following question.

Question 1.1 (Godsil-Meagher). Let $G = \text{Sym}(n)$ and H be a multiplicity-free subgroup of G , and define $\Omega = G/H$. Assume that $\lambda \vdash n$ is the second largest in dominance ordering in $\Lambda(n, H)$ and $\{S, \Omega \setminus S\}$ is the orbit partition of $\text{Sym}(\lambda)$ on Ω .

(a) Is there an orbital graph of G acting on G/H in which S is a coclique?

- (b) Is the eigenvalue corresponding to the λ -module the least eigenvalue for such graphs?

1.1 Motivation

Our motivation to study this question is closely related to a famous extremal set theory theorem. The Erdős-Ko-Rado (EKR) theorem is one of the most important results in extremal combinatorics. We say that a collection of k -subsets of $[n] := \{1, 2, \dots, n\}$ is a t -intersecting family if $|A \cap B| \geq t$ for all $A, B \in \mathcal{F}$. The EKR theorem is stated as follows.

Theorem 1.2 (EKR [8]). *For any two positive integers $k \geq t$, there exists $n_0(k, t)$ such that if $n \geq n_0(k, t)$ and \mathcal{F} is a t -intersecting family of k -subsets of $[n] := \{1, 2, \dots, n\}$, then $|\mathcal{F}| \leq \binom{n-t}{k-t}$. In addition, equality holds if and only if there exists $S \subset [n]$ of size t such that*

$$\mathcal{F} = \{A \subset [n] : |A| = k \text{ and } S \subset A\}.$$

For the case where $t = 1$, Erdős, Ko and Rado also proved that $n_0(k, 1) = 2k + 1$.

The EKR theorem has been extensively studied and extended to other objects such as vector spaces [14] and permutations [7]. There are various proofs of the EKR theorem which range from purely combinatorial [8], to probabilistic [14] and algebraic [19].

We now exhibit the relations between a certain association scheme and the EKR theorem. Let $n \geq 2k$ be two positive integers and define $\binom{[n]}{k}$ to be the collection of all k -subsets of $[n]$. For any $0 \leq i \leq k$, define

$$\mathcal{O}_i = \left\{ (A, B) \in \binom{[n]}{k} \times \binom{[n]}{k} : |A \cap B| = k - i \right\}.$$

The *Johnson scheme* $\mathcal{J}(n, k)$ is the association scheme given by

$$\left(\binom{[n]}{k}, \{\mathcal{O}_0, \mathcal{O}_1, \dots, \mathcal{O}_k\} \right).$$

The Johnson scheme contains the *Johnson graph* $J(n, k)$ and the *Kneser graph* $K(n, k)$, which are the orbital graphs of \mathcal{O}_1 and \mathcal{O}_k , respectively. The Kneser graph is important in the study of the EKR theorem since it encodes the 1-intersecting sets of k -subsets of $[n]$. Given a *co clique* (or independent set) \mathcal{S} of $K(n, k)$, it is not hard to check that \mathcal{S} has the property that $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{S}$, i.e., it is 1-intersecting. Conversely, any collection of k -subsets of $[n]$ with the property that any two elements intersect is a co clique of $K(n, k)$. Hence, a collection \mathcal{F} of k -subsets of $[n]$ is a maximum 1-intersecting family if and only if it is a maximum

coclique in the Kneser graph $K(n, k)$. Using the Johnson scheme, Wilson [19] also gave an algebraic proof of the EKR theorem and described precisely the smallest bound $n_0(k, t)$ in Theorem 1.2 for which the results of the EKR theorem hold.

One important aspect of the Johnson scheme is that it is schurian, that is, it arises from the orbital scheme of a transitive group. The corresponding group action for the case of the Johnson scheme is $\text{Sym}(n)$ acting on the k -subsets of $[n]$, or equivalently, on the cosets of $\text{Sym}(k) \times \text{Sym}(n - k)$. Another important property of $\text{Sym}(k) \times \text{Sym}(n - k)$ is that it is a multiplicity-free subgroup of $\text{Sym}(n)$. Hence, the Johnson scheme $\mathcal{J}(n, k)$ is a commutative association scheme. The second largest partition in dominance ordering in the corresponding permutation character is $[n - 1, 1]$. Further, one of the two orbits of the Young subgroup $\text{Sym}([n - 1, 1])$ is a maximum coclique in the Kneser graph $K(n, k)$ and its smallest eigenvalue is afforded by the $[n - 1, 1]$ -module. In other words, Question 1.1 is true for the Gelfand pair $(G, H) = (\text{Sym}(n), \text{Sym}(k) \times \text{Sym}(n - k))$.

Now, let $(G = \text{Sym}(n), H)$ be a Gelfand pair for which $\lambda \in \Lambda(n, H)$ is the second largest in dominance ordering. If Question 1.1 is true for (G, H) , then we obtain an EKR type theorem on G/H or on the corresponding combinatorial objects as follows. Since Question 1.1 (a) is true, there exists an orbital graph in which an orbit of $\text{Sym}(\lambda)$ acting on the cosets G/H is a coclique. Let $\mathcal{K}(G, H)$ be the union of all orbital graphs with this property. Let us consider the following terminologies.

Definition 1.3.

- (1) Two cosets xH and yH of H are (G, H) -intersecting if they are not adjacent in $\mathcal{K}(G, H)$.
- (2) A collection \mathcal{F} of cosets in G/H is (G, H) -intersecting if any pair of its elements are (G, H) -intersecting.
- (3) The orbit of a conjugate of $\text{Sym}(\lambda)$ on G/H which is a (G, H) -intersecting family is called a (G, H) -canonical intersecting family.

As Question 1.1 (b) is also true, one can prove that any maximum (G, H) -intersecting family (i.e., a maximum coclique in $\mathcal{K}(G, H)$) has size equal to the size of the (G, H) -intersecting orbit of $\text{Sym}(\lambda)$. Using these terminologies, one can pose the following question.

Problem 1.4. Assume that (G, H) is a Gelfand pair for which Question 1.1 is true. Are the (G, H) -canonical intersecting families the only (G, H) -intersecting families of maximum size?

If the answer to Problem 1.4 is true, then we obtain a full EKR theorem for the cosets G/H in the sense that the (G, H) -intersecting families have size at most the size of a (G, H) -canonical intersecting family, and those attaining this bound must be a (G, H) -canonical intersecting family.

Line	Group		n	Index	Rank
1	$\text{Sym}(k) \times \text{Sym}(n - k)$	$(2k \leq n)$	n	$\binom{n}{k}$	$k + 1$
2	$\text{Alt}(k) \times \text{Sym}(n - k)$	$(2k \leq n, k \neq 2)$	n	$2\binom{n}{k}$	$k + 3$
3	$\text{Sym}(k) \times \text{Alt}(n - k)$	$(2k \leq n, k \neq n - 2)$	n	$2\binom{n}{k}$	$k + 3$
4	$\text{Alt}(k) \times \text{Alt}(n - k)$	$(k \geq 3, 2k \leq n - 2)$	n	$4\binom{n}{k}$	$2k + 6$
5	$(\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)$	$(2k \leq n, (n, k) \neq (4, 2))$	n	$2\binom{n}{k}$	$2k + 2$
6	$(\text{Sym}(2) \wr \text{Sym}(k)) \times \text{Sym}(1)$		$2k + 1$		

Table 1: Multiplicity-free intransitive groups

1.2 Main results

For the case where $G = \text{Sym}(2k)$ and $H = \text{Sym}(2) \wr \text{Sym}(k)$, the association scheme corresponding to the Gelfand pair (G, H) is the *perfect matching scheme* of the complete graph K_{2k} (see [11] for details on this). An EKR type theorem on perfect matchings of the complete graph K_{2n} was proved by Godsil and Meagher in [10], and also Lindzey in [15]. The main technique used in the proof of this EKR type theorem can be extended to prove that the answer to Question 1.1 for the Gelfand pair $(\text{Sym}(2k), \text{Sym}(2) \wr \text{Sym}(k))$ is affirmative.

The answer to Question 1.1 is however not always affirmative. The Gelfand pair $(G, H) = (\text{Sym}(8), \text{Sym}(4) \wr \text{Sym}(2))$ induces a commutative rank 3 association scheme. The character table of this scheme is

$$\left(\begin{array}{ccc|c} 1 & 16 & 18 & 1 \\ 1 & -4 & 3 & 14 \\ 1 & 2 & -3 & 20 \end{array} \right).$$

Using the Ratio Bound (see Theorem 2.5), it is not hard to see that a coclique in either of the two non-trivial orbital graphs is of size at most 7. Moreover, since $\Lambda(8, H) = \{[8], [6, 2], [4, 4]\}$, the second largest in dominance ordering is $[6, 2]$ which induces two orbits of size 15 and 20. Consequently, we have a negative answer to Question 1.1. Our first result, which is a generalization of this example, is stated as follows.

Theorem 1.5. *The answer to Question 1.1 is negative for the multiplicity-free subgroup $\text{Sym}(k) \wr \text{Sym}(2)$ of $\text{Sym}(2k)$.*

In this paper, we initiate the study of Question 1.1 by considering the Gelfand pair $(\text{Sym}(n), H)$ for which the second largest partition in dominance ordering is $[n - 1, 1]$. The multiplicity-free subgroups of $\text{Sym}(n)$ were classified in [9]. We will focus on the families of multiplicity-free subgroups of $\text{Sym}(n)$ given in Table 1 in this work.

We state our next main result.

Theorem 1.6. *The answer to Question 1.1 is affirmative for the Gelfand pairs $(\text{Sym}(n), H)$, where H is the group in lines 2-6 of Table 1.*

1.3 Structure of the paper

This paper is organized as follows. In Sections 2 and 3, we give some background results on permutation groups, spectral graph theory techniques and association schemes. In Section 4, we give a proof of Theorem 1.5. In Section 5, we review the proof that Question 1.1 is true for $(\text{Sym}(n), \text{Sym}(k) \times \text{Sym}(n - k))$. The proof of Theorem 1.6 is spread in Sections 6, 7, 8, and 9. We conclude this paper by stating some interesting questions and problems in Section 10.

2 Background results

2.1 Permutation groups

Let G be a group and $H \leq G$ be a subgroup. The group G acts on G/H through $g(xH) = gxH$, for any $g \in G$ and $xH \in G/H$. It is not hard to see that this action is transitive since for any $xH, yH \in G/H$, we have $(yx^{-1})(xH) = yH$. We will denote the stabilizer of xH in this action of G on G/H by $\text{Stab}(G, xH)$. It is clear that $\text{Stab}(G, H) = H$. It is well known that any finite transitive group $G \leq \text{Sym}(\Omega)$ is permutation equivalent to G in its action on G/H , where H is the stabilizer of any $\omega \in \Omega$. This correspondence enables us to switch between the cosets or a set of specific combinatorial objects, whichever is easier. For any $\omega \in \Omega$, we let $\text{Stab}(G, \omega)$ be the stabilizer of ω in G . Since transitive permutation groups are faithful, we will always assume that H is a core-free subgroup of G .

The transitive group G acting on G/H induces an action on $G/H \times G/H$ by componentwise multiplication. The orbits of this action are called the *orbitals* of G and the *rank* of G is the number of orbitals. We note that if (xH, yH) belongs to an orbital \mathcal{O} , then

$$\mathcal{O} = G.(xH, yH) = \{(gxH, gyH) : g \in G\}.$$

By transitivity of G , the set $\{(xH, xH) : xH \in G/H\}$ is always an orbital of G . Other orbitals of G must partition the set $\{(xH, yH) : xH \neq yH \in G/H\}$. If \mathcal{O} is an orbital of G such that $\mathcal{O} = \{(yH, xH) : (xH, yH) \in \mathcal{O}\}$, then we say that \mathcal{O} is *self-paired*, otherwise, it is called *non-self-paired*.

The next lemma gives a well-known correspondence between orbitals and *suborbits* (i.e., orbits of a point stabilizer). Its proof can be found in any standard textbook on permutation groups.

Lemma 2.1. *The map which takes any orbital \mathcal{O} to the suborbit $\{xH : (H, xH) \in \mathcal{O}\}$ is a bijection.*

For any $g \in G$, define

$$\text{fix}_H^G(g) = |\{xH \in G/H : g(xH) = xH\}|. \quad (2.1)$$

The *permutation character* of the action of G on G/H is the character $\text{fix}_H^G : G \rightarrow \mathbb{C}$ which takes any $g \in G$ to $\text{fix}_H^G(g)$.

The next lemma gives the number of orbitals.

Lemma 2.2. *The number of orbitals of G acting on G/H is*

$$\frac{1}{|G|} \sum_{g \in G} \text{fix}_H^G(g)^2.$$

We will denote the set of complex irreducible characters of G by $\text{Irr}(G)$. Recall that given $\phi, \psi \in \text{Irr}(G)$, a natural inner product between ϕ and ψ is given by

$$\langle \phi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \phi(g) \overline{\psi(g)}.$$

It is not hard to see that, in fact, $\text{fix}_H^G = \mathbf{1}_H^G$. Moreover, the trivial character $\mathbf{1}_G$ of G is always a constituent of $\mathbf{1}_H^G$ and its multiplicity is equal to 1 since

$$\langle \mathbf{1}_H^G, \mathbf{1}_G \rangle = \frac{1}{|G|} \sum_{g \in G} \text{fix}_H^G(g) = 1.$$

Using Lemma 2.2, it is easy to see that the rank of the transitive group G (or the corresponding association scheme) is $\langle \mathbf{1}_H^G, \mathbf{1}_H^G \rangle$.

2.2 Equitable partitions

Let $X = (V, E)$ be a graph. A partition $\pi = \{V_1, V_2, \dots, V_k\}$ of the vertex set of X is called *equitable* if for any $1 \leq i, j \leq k$ there exists a number a_{ij} such that the number of neighbours in V_j of any vertex of V_i is equal to a_{ij} . If π is an equitable partition of X , then the *quotient graph* X/π is the directed multigraph with vertex set equal to the elements of π and for any $i, j \in \{1, 2, \dots, k\}$, there are exactly a_{ij} arcs from V_i to V_j . The *quotient matrix* of an equitable partition π is the $k \times k$ matrix $A(X/\pi)$ indexed in its rows and columns by $\{1, 2, \dots, k\}$ and whose ij -entry is a_{ij} .

Equitable partitions are important in spectral graph theory due to the following result.

Theorem 2.3. [11, Lemma 2.2.2] *If π is an equitable partition of X , then the characteristic polynomial of $A(X/\pi)$ divides the characteristic polynomial of the adjacency matrix of X . In particular, an eigenvalue of $A(X/\pi)$ is an eigenvalue of X .*

The above theorem enables one to determine certain eigenvalues of the graph X through X/π . Given an eigenvalue of $A(X/\pi)$ and a corresponding eigenvector v , one can lift v into an eigenvector of X . For any equitable partition $\pi = \{V_1, V_2, \dots, V_k\}$, define the $|V(X)| \times k$ matrix P_π whose rows and columns are respectively indexed by vertices of X and the partition π , and whose (x, V_j) -entry is equal to 1 if $x \in V_j$ and 0 otherwise. The matrix P_π is called the *characteristic matrix* of π . The next lemma shows the importance of the matrix P_π .

Lemma 2.4. *If v is an eigenvector of $A(X/\pi)$ corresponding to the eigenvalue θ , then $P_\pi v$ is an eigenvector of X with the same eigenvalue.*

The proof of this lemma is given in [11, Lemma 2.2.1].

2.3 Hoffman’s Ratio Bound

We recall the well-known Ratio Bound which is due to Hoffman. Let $X = (V, E)$ be a k -regular graph. It is well known that k is an eigenvalue of X . The vector of all-ones, denoted by $\mathbf{1}$, is an eigenvector of X corresponding to the eigenvalue k and it is an easy exercise to prove that the multiplicity of k is the number of components of X . For any $S \subset V(X)$, the *characteristic vector* of S is the vector $v_S \in \mathbb{R}^{|V(X)|}$ which is indexed by vertices of X and such that the x -entry of v_S is equal to 1 if $x \in S$, and 0 otherwise.

We state the Ratio Bound in the next theorem.

Theorem 2.5 (Ratio Bound). *If $X = (V, E)$ is a k -regular graph with least eigenvalue τ , then the independence number of X satisfies*

$$\alpha(X) \leq \frac{|V(X)|}{1 - \frac{k}{\tau}}.$$

A set S is a coclique for which equality holds if and only if $v_S - \frac{|S|}{|V(X)|}\mathbf{1}$ is an eigenvector of the eigenvalue τ .

A proof and the history about the Ratio Bound can be found in [12].

2.4 Graph products

In this subsection, we will recall some important graph products that are used later in this work.

Definition 2.6. Let $X = (V(X), E(X))$ and $Y = (V(Y), E(Y))$ be two graphs. The direct product $X \times Y$ of the graphs X and Y is the graph whose vertex set is $V(X) \times V(Y)$ and

$$(x, y) \sim_{X \times Y} (u, v) \Leftrightarrow x \sim_X u \text{ and } y \sim_Y v.$$

Definition 2.7. Let $X = (V(X), E(X))$ and $Y = (V(Y), E(Y))$ be two graphs. The strong product $X \boxtimes Y$ of the graphs X and Y is the graph whose vertex set is $V(X) \times V(Y)$ and

$$(x, y) \sim_{X \boxtimes Y} (u, v) \Leftrightarrow \begin{cases} x = u \text{ and } y \sim_Y v \\ y = v \text{ and } x \sim_X u \\ x \sim_X u \text{ and } y \sim_Y v. \end{cases}$$

For any $n \geq 2$, let I_n be the $n \times n$ identity matrix and J_n be the $n \times n$ matrix whose entries consist of 1. The next proposition gives the adjacency matrices of the graph products defined above.

Proposition 2.8. *Let X and Y be two graphs with adjacency matrices A and B , respectively.*

- (a) *The adjacency matrix of $X \times Y$ is $A \otimes B$. In particular, the eigenvalues of $X \times Y$ are of the form ab , where a and b are eigenvalues of A and B , respectively.*
- (b) *The adjacency matrix of $X \boxtimes Y$ is $(A + I_{|V(X)|}) \otimes (B + I_{|V(Y)|}) - I_{|V(X)||V(Y)|}$. In particular, the eigenvalues of $X \boxtimes Y$ are of the form $(1 + a)(1 + b) - 1$.*

Let us now introduce a new graph that is crucial to the main results of this paper.

Definition 2.9. Let $X = (V(X), E(X))$ be a graph. The graph product $X \bowtie K_2$ is defined to be the graph obtained by taking two disjoint copies of X , and a vertex from one copy is adjacent to a vertex from the other copy if they are adjacent in $X \times K_2$.

Note that the graph $X \bowtie K_2$ is exactly the graph $X \boxtimes K_2$ in which the perfect matching $\{(x, 1) \sim (x, 0) : x \in V(X)\}$ is removed.

Proposition 2.10. *Let X be a graph with adjacency matrix A . The adjacency matrix of $X \bowtie K_2$ is $J_2 \otimes A$.*

Proof. Let A be the adjacency matrix of $X = (V, E)$ and assume that $|V| = n$. The adjacency matrix of $X \bowtie K_2$ can be viewed as a block matrix, whose blocks are indexed by V . Hence, the matrix is

$$\begin{bmatrix} A & A \\ A & A \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes A.$$

□

3 The association scheme arising from a Gelfand pair $(\text{Sym}(n), H)$

3.1 Eigenvalues

Let G be a group and $H \leq G$ be a multiplicity-free subgroup. In this subsection, we recall the formula giving the eigenvalues of each digraph in the association scheme arising from the Gelfand pair (G, H) . Assume that $k = \langle \mathbf{1}_H^G, \mathbf{1}_H^G \rangle$ and let $\mathcal{O} = \{O_0, O_1, \dots, O_{k-1}\}$ be the set of all orbitals of G acting on G/H . For any $0 \leq i \leq$

$k - 1$, let X_i be the orbital digraph determined by O_i and let $A(X_i)$ be its adjacency matrix. Define

$$\Lambda(G, H) := \{ \phi \in \text{Irr}(G) : \langle \mathbf{1}_H^G, \phi \rangle = 1 \}.$$

For any $0 \leq i \leq k - 1$, let

$$k_i := | \{ yH \in G/H : (H, yH) \in O_i \} |. \tag{3.1}$$

The next theorem gives the eigenvalues of X_i for $0 \leq i \leq k - 1$. Its proof is given in [11, Lemma 13.8.3] or [1, Theorem 11.10].

Theorem 3.1. *The eigenvalues of X_i are uniquely determined by the irreducible characters in $\Lambda(G, H)$. For any $\phi \in \Lambda(G, H)$, the eigenvalue of X_i afforded by ϕ is given by*

$$\xi_\phi(X_i) := \frac{k_i}{|H|} \sum_{h \in H} \phi(x_\ell h),$$

where $x_\ell \in G$ such that $(H, x_\ell H) \in O_i$.

The formula for the eigenvalues given in the previous theorem is usually hard to manipulate since there is no way to control the elements of the coset $x_\ell H$, which in turn makes the corresponding character value difficult to determine. It is sometimes possible to compute certain eigenvalues via special techniques. In the next subsection, we will compute certain eigenvalues using equitable partitions for the case when $G = \text{Sym}(n)$.

Let $G = \text{Sym}(n)$. It is well known that the representations of G are indexed by the partitions of the integer n . For any $\lambda \vdash n$, the corresponding irreducible $\mathbb{C}G$ -module is the *Specht module* S^λ and we denote its irreducible character by χ^λ . If (G, H) is a Gelfand pair, then the eigenvalues of a graph in the corresponding orbital scheme are indexed by the partitions in $\Lambda(n, H)$. For any $\lambda \in \Lambda(n, H)$, we let ξ_λ be the eigenvalue afforded by χ^λ . In addition, we will refer to the irreducible $\mathbb{C}G$ -module S^λ as the λ -*module*.

An important property of the symmetric group is that all its irreducible characters are realized over \mathbb{Z} . Therefore, the eigenvalues of an orbital graph arising from a Gelfand pair $(\text{Sym}(n), H)$ are rational. As an orbital scheme arising from the action of G on the cosets G/H is symmetric if and only if (G, H) is a Gelfand pair and each irreducible constituent (of the permutation character) is real-valued, we conclude the following proposition.

Proposition 3.2. *If $(\text{Sym}(n), H)$ is a Gelfand pair, then every orbital digraph from the orbital scheme of $(\text{Sym}(n), H)$ is an undirected graph.*

3.2 Equitable partitions

We assume henceforth that $G = \text{Sym}(n)$ and let $H \leq G$ such that (G, H) is a Gelfand pair. Let k be the rank of the group G acting on G/H . Recall that $\mathcal{O} =$

$\{O_0, O_1, \dots, O_{k-1}\}$ is the set of all orbitals of G acting on G/H and for any $0 \leq i \leq k - 1$, the graph X_i is the orbital graph determined by O_i and the matrix $A(X_i)$ is its adjacency matrix. Recall also that k_i is the degree of the orbital graph X_i , for $0 \leq i \leq k - 1$ (see (3.1)). Let $\lambda \vdash n$ be the partition which is the second largest in dominance ordering in $\Lambda(n, H)$. Let $K = \text{Sym}(\lambda)$. We state the following theorem whose proof is given in [11, Theorem 13.9.1].

Lemma 3.3. *The action of K on G/H by left multiplication admits exactly two orbits.*

Using these orbits, we can sometimes easily compute certain eigenvalues of the graphs X_i defined in the previous section. Let $\Omega = G/H$. First, we note that the set of orbits $\pi = \{\mathcal{S}, \Omega \setminus \mathcal{S}\}$ forms a partition of Ω which is equitable, for all orbital graphs in the orbital scheme. The quotient matrix of each orbital graph in the orbital scheme is given in the next lemma.

Lemma 3.4. *For any $0 \leq i \leq k - 1$, there exists a number $0 \leq a_i \leq k_i - 1$ such that the quotient matrix corresponding to X_i/π is*

$$A(X_i/\pi) = \begin{bmatrix} a_i & k_i - a_i \\ \frac{(k_i - a_i)|\mathcal{S}|}{|\Omega| - |\mathcal{S}|} & \frac{k_i|\Omega| + a_i|\mathcal{S}| - 2k_i|\mathcal{S}|}{|\Omega| - |\mathcal{S}|} \end{bmatrix}.$$

Proof. Since π is equitable with two parts, let a_i be the number of vertices in \mathcal{S} adjacent to a given vertex in \mathcal{S} . If the rows and columns of $A(X_i/\pi)$ are arranged with respect to $\{\mathcal{S}, \Omega \setminus \mathcal{S}\}$, then it is clear that $A(X_i/\pi)_{11} = a_i$ and $A(X_i/\pi)_{12} = k_i - a_i$, by regularity of X_i . The entry $A(X_i/\pi)_{21}$ can be computed using double counting. Consider the set

$$V = \{(xH, yH) : xH \in \Omega \setminus \mathcal{S} \text{ and } yH \in \mathcal{S}\}.$$

On the one hand, given $xH \in \Omega \setminus \mathcal{S}$ we have

$$A(X_i/\pi)_{21} = |\{(xH, yH) : yH \in \mathcal{S}\}|.$$

Moreover,

$$|V| = \left| \bigcup_{xH \in \Omega \setminus \mathcal{S}} \{(xH, yH) : yH \in \mathcal{S}\} \right| = A(X_i/\pi)_{12}|\Omega \setminus \mathcal{S}| = A(X_i/\pi)_{21} (|\Omega| - |\mathcal{S}|).$$

On the other hand, we have

$$|V| = \left| \bigcup_{yH \in \mathcal{S}} \{(xH, yH) : xH \in \Omega \setminus \mathcal{S}\} \right| = |\mathcal{S}|A(X_i/\pi)_{12} = |\mathcal{S}|(k_i - a_i).$$

Therefore, we have $A(X_i/\pi)_{21} = \frac{|\mathcal{S}|(k_i - a_i)}{|\Omega| - |\mathcal{S}|}$. The entry $A(X_i/\pi)_{22}$ can be easily deduced since it is equal to $k_i - A(X_i/\pi)_{21}$. \square

By Theorem 2.2, the eigenvalues of $A(X_i/\pi)$ are also eigenvalues of the graph X_i , for any $0 \leq i \leq k - 1$. The eigenvalues of $A(X_i/\pi)$ are

$$k_i \text{ and } a_i - \frac{(k_i - a_i)|\mathcal{S}|}{|\Omega| - |\mathcal{S}|}.$$

We now present some consequences of Lemma 3.4 that are crucial to the main result of this paper.

Lemma 3.5. *Let $\Omega = G/H$ and $\pi = \{\mathcal{S}, \Omega \setminus \mathcal{S}\}$ be the set of orbits of $K = \text{Sym}(\lambda)$. For $0 \leq i \leq k - 1$, the set \mathcal{S} is a coclique of X_i if and only if $a_i = 0$.*

The proof of this lemma is straightforward, thus it is omitted. The next lemma is an easy exercise from [11, Exercise 13.7].

Lemma 3.6. *Let $\lambda \vdash n$ and let X be an orbital graph of the transitive action of $\text{Sym}(n)$ on Ω . If π is the orbits partition of $\text{Sym}(\lambda)$ on Ω , then an eigenvalue η of $A(X/\pi)$ belongs to a certain μ -module, where $\lambda \trianglelefteq \mu$.*

The next corollary gives more details on the relation between cocliques from orbit partitions and the λ -module, whenever $\lambda = [n - 1, 1]$ is the second largest in dominance ordering in $\Lambda(n, H)$. Its proof follows immediately from the above lemma.

Corollary 3.7. *Let $G = \text{Sym}(n)$ and (G, H) be a Gelfand pair such that $[n - 1, 1]$ is the second largest in dominance ordering in $\Lambda(n, H)$. Let $\Omega = G/H$ and $\pi = \{\mathcal{S}, \Omega \setminus \mathcal{S}\}$ be the set of orbits of $K = \text{Sym}([n - 1, 1])$. For any $0 \leq i \leq k - 1$, if the set \mathcal{S} is a coclique of X_i , then $-\frac{k_i|\mathcal{S}|}{|\Omega| - |\mathcal{S}|}$ is the eigenvalue afforded by the $[n - 1, 1]$ -module.*

4 Proof of Theorem 1.5

Let $G = \text{Sym}(2k)$ and $H = \text{Sym}(k) \wr \text{Sym}(2)$. The rank of the group G acting on G/H is $\lfloor \frac{k}{2} \rfloor + 1$ since by [9], we have

$$\mathbf{1}_H^G = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \chi^{[2k-2i, 2i]}.$$

Therefore, the second largest in dominance ordering in $\Lambda(n, H)$ is $[2k - 2, 2]$. We note that the combinatorial objects that correspond to the cosets of H in G are the uniform partitions of $[2k]$ into 2 blocks of size k . That is, the collections

$$U_k := \{ \{B_1, B_2\} : B_1 \cup B_2 = [2k], B_1 \cap B_2 = \emptyset, |B_1| = |B_2| = k \}.$$

The action of the symmetric group $\text{Sym}(2k)$ on $[2k]$ induces an action on U_k . It is not hard to see that $H = \text{Sym}(k) \wr \text{Sym}(2)$ is the stabilizer of the partition $\{ \{1, 2, \dots, k\}, \{k + 1, k + 2, \dots, 2k\} \} \in U_k$. Moreover, this action is permutation equivalent to that of G on G/H since there is a unique conjugacy class of subgroups isomorphic to H .

For any $A \in \binom{[2k]}{k}$, we let \underline{A} be the complement of A in $[2k]$.

Lemma 4.1. *The orbitals of $G = \text{Sym}(2k)$ in its action on U_k are of the form*

$$\mathcal{O}_i = \{(\mathcal{B}, \mathcal{B}') \in U_k \times U_k : \text{there exists } B \in \mathcal{B}, B' \in \mathcal{B}' \text{ such that } |B \cap B'| = k - i\},$$

for $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$.

Proof. It is worth noting that if two partitions $\mathcal{B} = \{B, \underline{B}\}$ and $\mathcal{B}' = \{B', \underline{B}'\}$ of U_k are such that $|B \cap B'| = k - i$, then we must have $|B \cap \underline{B}'| = i$.

For any $0 \leq i \leq k$, let $X_i = \{1, 2, \dots, k - i, k + 1, k + 2, \dots, k + i\}$. Note that $X_0 = \{1, 2, \dots, k\}$ and $X_k = \{k + 1, k + 2, \dots, 2k\}$. Let $\ell = \lfloor \frac{k}{2} \rfloor$ and for $0 \leq i \leq \ell$, define $\mathcal{B}_i := \{X_i, \underline{X}_i\}$. We claim that $\{\mathcal{B}_i : 0 \leq i \leq \ell\}$ consists of a complete list of representatives from distinct suborbits of $\text{Stab}(\text{Sym}(2k), \mathcal{B}_0) = \text{Sym}(k) \wr \text{Sym}(2)$. It is clear that X_0 is in a single orbit of $\text{Sym}(k) \wr \text{Sym}(2)$. Assume that there exists $i < j$ in the set $\{1, \dots, \ell\}$ such that \mathcal{B}_i and \mathcal{B}_j are in the same orbit of $\text{Sym}(k) \wr \text{Sym}(2)$. If $\sigma \in \text{Sym}(k) \wr \text{Sym}(2)$ such that $\sigma(\mathcal{B}_i) = \mathcal{B}_j$, then either $\sigma(X_i) = X_j$ or $\sigma(X_i) = \underline{X}_j$.

Case 1: Assume that $\sigma(X_i) = X_j$. If $\sigma(X_0) = X_0$, then σ must map the elements $X_i \cap X_0 = \{1, 2, \dots, k - i\}$ to $X_j \cap X_0 = \{1, 2, \dots, k - j\}$. By the Pigeonhole Principle, this cannot happen since $i < j$. Similarly, if $\sigma(X_0) = \underline{X}_0 = X_k$, then σ has to map $\{k + 1, k + 2, \dots, k + i\}$ to $\{1, 2, \dots, k - j\}$. This is only possible if $k - j = i$. As i and j are distinct and at most $\ell \leq \frac{k}{2}$, we conclude this case also cannot happen.

Case 2: Assume that $\sigma(X_i) = \underline{X}_j$. Note that $\underline{X}_j = \{k - j + 1, \dots, k, k + j + 1, \dots, 2k\}$. By a careful analysis of the image of X_0 by σ , we also conclude that this case is not possible.

Consequently, the set $\{\mathcal{B}_i : 0 \leq i \leq \ell\}$ consists of elements in different orbits of $\text{Sym}(k) \wr \text{Sym}(2)$. Since the cardinality of this set and the rank of G acting on U_k coincide, we conclude that the orbitals of G on U_k are those listed in the statement of the lemma. We obtain the rest by making $\text{Sym}(2k)$ act on every representative of the orbitals. □

Using this lemma, we can show that the orbits of $K = \text{Sym}([2k - 2, 2])$ are not cocliques of any orbital graph in this association scheme. The orbits of K are \mathcal{S} and $U_k \setminus \mathcal{S}$, where

$$\mathcal{S} = \{\{B, \underline{B}\} : 2k - 1, 2k \in B\}.$$

In the next theorem, we claim that neither of \mathcal{S} nor $U_k \setminus \mathcal{S}$ is a coclique of a graph in the orbital scheme corresponding to $\text{Sym}(2k)$ acting on U_k .

Theorem 4.2. *The collection \mathcal{S} is not a coclique of the orbital graph corresponding to \mathcal{O}_i , for any $0 \leq i \leq \lfloor \frac{k}{2} \rfloor$.*

Proof. Let $1 \leq i < \ell = \lfloor \frac{k}{2} \rfloor$. In the orbital graph of \mathcal{O}_i , by definition, there is an edge between $\{B_1, B_2\}$ and $\{B'_1, B'_2\}$ if $|B_1 \cap B'_1| \in \{i, k - i\}$. Consider the sets

$$A = \{1, 2, \dots, k - i - 2, k + 1, k + 2, \dots, k + i, 2k - 1, 2k\}$$

$$B = \{1, 2, \dots, k - i - 2, k + i + 1, k + i + 2, \dots, k + 2i, 2k - 1, 2k\}.$$

Since $|A \cap B| = k - i$, the partitions $\{A, \underline{A}\}$ and $\{B, \underline{B}\}$ are adjacent in the orbital graph of \mathcal{O}_i . As these partitions are in \mathcal{S} , we conclude that \mathcal{S} is not a coclique of the orbital graph of \mathcal{O}_i , for any $1 \leq i < \ell$.

For the orbital graph \mathcal{O}_ℓ , the sets

$$A = \{k + 1, k + 2, \dots, k + \lfloor \frac{k}{2} \rfloor, k + \lfloor \frac{k}{2} \rfloor + 1, \dots, 2k - 1, 2k\} \in \mathcal{S}, \text{ and}$$

$$B = \{1, 2, \dots, \lfloor \frac{k}{2} \rfloor, k + \lfloor \frac{k}{2} \rfloor + 1, \dots, 2k - 1, 2k\} \in \mathcal{S}$$

are such that $|A \cap B| = k - \ell = \ell$. Therefore, $\{A, \underline{A}\}$ and $\{B, \underline{B}\}$ are adjacent in \mathcal{O}_ℓ . This completes the proof. \square

Remark 4.3. The orbital graph of \mathcal{O}_1 is called the folded Johnson graph $\bar{J}(2k, k)$ (see [5, Section 9.1] for details). This graph encodes certain combinatorial objects as its cocliques. Two partitions \mathcal{B} and \mathcal{B}' of U_k are called partially 2-intersecting if there exists $B \in \mathcal{B}$ and $B' \in \mathcal{B}'$ such that $|B \cap B'| \geq 2$. It is straightforward to verify that a maximum partially 2-intersecting family of U_k is a coclique in the orbital graph of \mathcal{O}_1 (i.e., the folded Johnson graph $\bar{J}(2k, k)$), and vice versa. As far as we know, there is nothing known about the cocliques of this graph.

5 The orbital scheme from line 1 of Table 1

In this section, we will consider the schurian association scheme obtained from the action of $\text{Sym}(n)$ on the cosets of $\text{Sym}(k) \times \text{Sym}(n - k)$. We will exhibit an action on certain combinatorial objects that is permutation equivalent to the latter. To this end, we will recall a way of constructing the k -subsets of $[n]$ with an equivalence relation.

Let n be a positive integer and $2 \leq k \leq n$. Consider the set

$$\mathcal{I}_{n,k} := \{f : [k] \rightarrow [n] : f \text{ is injective}\}.$$

An element of $\mathcal{I}_{n,k}$ can be represented as a sequence of the form (a_1, a_2, \dots, a_k) . Define the relation \mathcal{R} on $\mathcal{I}_{n,k}$ such that $(a_1, a_2, \dots, a_k) \mathcal{R} (b_1, b_2, \dots, b_k)$ if there exists $\sigma \in \text{Sym}(k)$ such that

$$a_i = b_{\sigma(i)}, \text{ for } i \in \{1, 2, \dots, k\}.$$

It is not hard to prove that the relation \mathcal{R} is in fact an equivalence relation. The equivalence classes of \mathcal{R} are of the form

$$\overline{(a_1, a_2, \dots, a_k)} = \{(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k)}) \in \mathcal{I}_{n,k} : \sigma \in \text{Sym}(k)\}.$$

These equivalence classes are naturally identified (or in bijection) with the k -subsets of $[n]$. We will denote the equivalence class $\overline{(a_1, a_2, \dots, a_k)}$ by $\{a_1, a_2, \dots, a_k\}$ from now on. We will also define $\binom{[n]}{k}$ to be the set of all k -subsets of $[n]$.

Given a k -subset $A = \{a_1, a_2, \dots, a_k\}$ of $[n]$, it is clear that the setwise stabilizer of A in the symmetric group $\text{Sym}(n)$ is equal to $\text{Sym}(A) \times \text{Sym}([n] \setminus A) \cong \text{Sym}(k) \times$

$\text{Sym}(n - k)$. In fact, it is not hard to see that the action of $\text{Sym}(n)$ on the cosets of $\text{Sym}(k) \times \text{Sym}(n - k)$ is permutation equivalent to the action of $\text{Sym}(n)$ on the k -subsets of $[n]$.

The association scheme which arises from the action of $\text{Sym}(n)$ on the k -subsets of $[n]$ is well known and well studied. The orbitals of this action are

$$\mathcal{O}_i = \left\{ (A, B) : A, B \in \binom{[n]}{k} \text{ such that } |A \cap B| = k - i \right\},$$

for $0 \leq i \leq k$. The corresponding association scheme is the *Johnson scheme* $\mathcal{J}(n, k)$. The permutation character corresponding to the action of $\text{Sym}(n)$ on the k -subsets is given by

$$\mathbf{1}_{\text{Sym}(k) \times \text{Sym}(n-k)}^{\text{Sym}(n)} = \sum_{i=0}^k \chi^{[n-i, i]}.$$

Hence, the second largest partition of $\Lambda(n, \text{Sym}(k) \times \text{Sym}(n - k))$ is equal to $[n - 1, 1]$. The orbits of the Young subgroup $\text{Sym}([n - 1, 1])$ on the k -subsets are S and $\binom{[n]}{k} \setminus S$, where

$$S = \left\{ A \in \binom{[n]}{k} : n \in A \right\}. \tag{5.1}$$

The graph obtained from the orbital \mathcal{O}_k has the property that two k -subsets of $[n]$, A and B , are adjacent if $A \cap B = \emptyset$. This graph is known as the *Kneser graph* $K(n, k)$. By the well-known Erdős-Ko-Rado theorem, the set S in (5.1) is a maximum coclique of $K(n, k)$. It is also well known that the eigenvalue afforded by the $[n - 1, 1]$ -module is the least eigenvalue of $K(n, k)$, which is equal to

$$-\frac{\binom{n-k}{k} \binom{n-1}{k-1}}{\binom{n-1}{k}}.$$

6 The orbital scheme from line 2 and line 3 of Table 1

In this section, we prove that the answer to Question 1.1 is affirmative for the Gelfand pair (G, H) , where $G = \text{Sym}(n)$ and H is a subgroup belonging to the set $\{\text{Alt}(k) \times \text{Sym}(n - k), \text{Sym}(k) \times \text{Alt}(n - k)\}$. We will only give the proof for the case $H = \text{Alt}(k) \times \text{Sym}(n - k)$ since the other case is similar.

6.1 The quasi k -subsets of $[n]$

Let n be a positive integer and $k \leq n$. Recall that $\mathcal{I}_{n,k}$ is the set of all injective maps from $[k]$ to $[n]$. Define the relation on $\mathcal{I}_{n,k}$ by

$$(a_1, a_2, \dots, a_k) \mathcal{R} (b_1, b_2, \dots, b_k) \Leftrightarrow \exists \sigma \in \text{Alt}(k) \text{ such that } b_i = a_{\sigma(i)},$$

for all $i \in \{1, 2, \dots, k\}$. It is clear that \mathcal{R} is an equivalence relation on $\mathcal{I}_{n,k}$. For any $(a_1, a_2, \dots, a_k) \in \mathcal{I}_{n,k}$, define the corresponding equivalence class of \mathcal{R} to be

$$\overline{(a_1, a_2, \dots, a_k)} := \{(a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k)}) : \sigma \in \text{Alt}(k)\}.$$

Next, let us compute the number of equivalence classes of \mathcal{R} . We claim that each k -subset $\{a_1, a_2, \dots, a_k\}$ determines two equivalence classes of \mathcal{R} . To see this, let T be the set of all injective maps from $[k]$ to $A = \{a_1, a_2, \dots, a_k\}$ (i.e., bijections). The group $\text{Sym}(k)$ acts naturally on T by permuting the entries. By the orbit-counting lemma on the action of $\text{Alt}(k)$ on T , we know that

$$\frac{2}{k!} \sum_{\sigma \in \text{Alt}(k)} |\{(b_1, b_2, \dots, b_k) \in T : b_{\sigma(i)} = b_i, \forall i \in \{1, 2, \dots, k\}\}| = \frac{2}{k!} k! = 2.$$

In other words, the alternating group $\text{Alt}(k)$ admits two orbits on T . The two equivalence classes of \mathcal{R} determined by $A = \{a_1, a_2, \dots, a_k\}$ with $a_1 < a_2 < \dots < a_k$ are

$$\{a_1, a_2, a_3 \dots, a_k\}^+ := \overline{(a_1, a_2, a_3 \dots, a_k)}$$

and

$$\{a_1, a_2, a_3 \dots, a_k\}^- := \overline{(a_2, a_1, a_3, \dots, a_k)}.$$

We will call the collection of all equivalence classes of \mathcal{R} the *quasi k -subset* of $[n]$ and we will denote it by $\binom{[n]}{k}^\pm$. The classes $\{a_1, a_2, a_3 \dots, a_k\}^+$ and $\{a_1, a_2, a_3 \dots, a_k\}^-$ are called the even and odd quasi k -subsets of $\{a_1, a_2, a_3 \dots, a_k\}$, respectively. Note that the number of quasi k -subsets of $[n]$ is $2\binom{[n]}{k}$.

6.2 Action of the $\text{Sym}(n)$ on quasi k -subsets

Let $n \geq 3$. Given a k -tuple $A = (a_1, a_2, \dots, a_k) \in \mathcal{I}_{n,k}$ and $\sigma \in \text{Sym}(n)$, the permutation σ acts naturally on A via $\sigma(A) := (\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))$. Assume that the elements of A are ordered in an increasing way, i.e., $a_1 < a_2 < \dots < a_k$.

- Assume $\sigma(a_{i_1}) < \sigma(a_{i_2}) < \dots < \sigma(a_{i_k})$. There is of course a unique permutation $\rho = \rho(\sigma, A)$ of $\{1, 2, \dots, k\}$ that reorders $(\sigma(a_1), \sigma(a_2), \dots, \sigma(a_k))$ in an increasing order by permuting the indices. More precisely, $\rho \in \text{Sym}(k)$ is the unique permutation such that

$$\rho(1) = i_1, \rho(2) = i_2, \dots, \rho(k) = i_k.$$

- For any $\sigma \in \text{Sym}(n)$ and $A = (a_1, a_2, \dots, a_k)$, define

$$\sigma(A^*) = \begin{cases} \sigma(A)^+ & \text{if } * = + \text{ and } \rho(\sigma, A) \in \text{Alt}(k) \\ \sigma(A)^- & \text{if } * = - \text{ and } \rho(\sigma, A) \in \text{Alt}(k) \\ \sigma(A)^- & \text{if } * = + \text{ and } \rho(\sigma, A) \notin \text{Alt}(k) \\ \sigma(A)^+ & \text{if } * = - \text{ and } \rho(\sigma, A) \notin \text{Alt}(k). \end{cases} \tag{6.1}$$

This yields an action of $\text{Sym}(n)$ on the quasi k -subsets of $[n]$.

Let $A = \{1, 2, \dots, k\}$ and let us determine the stabilizer of A^+ for this action of $\text{Sym}(n)$. If $\sigma \in \text{Sym}(n)$ fixes A^+ , then it must fix the set $A = \{1, 2, \dots, k\}$ setwise. Hence, $\sigma \in \text{Sym}(k) \times \text{Sym}(n - k)$. By definition, we have $\sigma(A^+) = A^+$ if and only if $\rho = \rho(\sigma, A) \in \text{Alt}(k)$. By noting that $\rho = \sigma_{|\text{Alt}(k)}^{-1}$, it is clear that $\sigma_{|\text{Alt}(k)} \in \text{Alt}(k)$. Therefore,

$$\text{Stab}(\text{Sym}(n), A^+) = \text{Stab}(\text{Sym}(n), A^-) = \text{Alt}(k) \times \text{Sym}(n - k).$$

Using the fact that there is a unique conjugacy class of groups isomorphic to $\text{Alt}(k) \times \text{Sym}(n - k)$ in $\text{Sym}(n)$, the next result follows immediately.

Lemma 6.1. *The action of $\text{Sym}(n)$ on the cosets of $\text{Alt}(k) \times \text{Sym}(n - k)$ is permutation equivalent to the action of $\text{Sym}(n)$ on quasi k -subsets of $[n]$.*

6.3 The quasi Johnson scheme

In this subsection, we describe the association scheme obtained from the action of $\text{Sym}(n)$ on cosets of $\text{Alt}(k) \times \text{Sym}(n - k)$. We first characterize all orbitals of the action of $\text{Sym}(n)$ on quasi k -subsets of $[n]$.

Theorem 6.2. *An orbital of $G = \text{Sym}(n)$ acting on the quasi k -subsets of $[n]$ is one of the following types.*

1) $\mathcal{O}_0^+ := G.(A^+, A^+)$ where $A \in \binom{[n]}{k}$. This set is

$$\mathcal{O}_0^+ = \left\{ (A^+, A^+) : A \in \binom{[n]}{k} \right\} \cup \left\{ (A^-, A^-) : A \in \binom{[n]}{k} \right\}$$

2) $\mathcal{O}_0^- := G.(A^+, A^-)$, where $A \in \binom{[n]}{k}$. This set is

$$\mathcal{O}_0^- = \left\{ (A^+, A^-) : A \in \binom{[n]}{k} \right\} \cup \left\{ (A^-, A^+) : A \in \binom{[n]}{k} \right\}.$$

3) $\mathcal{O}_1^- := G.(A^+, B^-)$, where $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - 1$. This set is

$$\mathcal{O}_1^- = \left\{ (A^+, B^-), (A^-, B^+) : A, B \in \binom{[n]}{k}, |A \cap B| = k - 1 \right\}.$$

4) $\mathcal{O}_1^+ := G.(A^+, B^+)$, where $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - 1$. This set is

$$\mathcal{O}_1^+ = \left\{ (A^+, B^+), (A^-, B^-) : A, B \in \binom{[n]}{k}, |A \cap B| = k - 1 \right\}$$

5) $\mathcal{O}_i := G.(A^+, B^+)$, for $2 \leq i \leq k$, where $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - i$. This set is

$$\mathcal{O}_i = \left\{ (A^+, B^+), (A^+, B^-), (A^-, B^+), (A^-, B^-) : A, B \in \binom{[n]}{k}, |A \cap B| = k - i \right\}.$$

Proof. By [9], the permutation character of the group $\text{Sym}(n)$ acting on quasi k -subsets of $[n]$ is

$$\mathbf{1}_{\text{Alt}(k) \times \text{Sym}(n-k)}^{\text{Sym}(n)} = \sum_{i=0}^k \chi^{[n-i,i]} + \chi^{[n-k,1^k]} + \chi^{[n-k+1,1^{k-1}]}.$$

Hence, the number of orbitals of the corresponding group action is $k + 3$.

Let $X = \{1, 2, 3, \dots, k\}$. Consider some distinct elements $x_1 < x_2 < \dots < x_k$ of the set $[n] \setminus \{1, 2, \dots, k\}$. For any $1 \leq i \leq k$, we define

$$X_i = \{1, 2, \dots, k - i, x_1, x_2, \dots, x_i\}.$$

It is not hard to verify that (X^+, X^+) , (X^+, X_1^+) , (X^+, X_1^-) , (X^+, X^-) , and (X^+, X_i^+) , for $2 \leq i \leq k$, are elements belonging to the five types sets listed in the statement of the theorem. To prove the theorem, it is enough to prove that $S = \{X^+, X_1^+, X_1^-, X^-\} \cup \{X_2^+, X_3^+, \dots, X_k^+\}$ are in different suborbits of the subgroup $\text{Stab}(\text{Sym}(n), X^+) = \text{Alt}(k) \times \text{Sym}(n - k)$.

It is clear that X^+ cannot be in the same orbit of $\text{Stab}(\text{Sym}(n), X^+)$ as any other element of S . We claim that for any $A^*, B^\dagger \in S$, where $*, \dagger \in \{+, -\}$, such that $|A \cap X| \neq |B \cap X|$, there exist no $\sigma \in \text{Alt}(k) \times \text{Sym}(n - k)$ such that $\sigma(A^*) = B^\dagger$. This is clear since assuming that $|A \cap X| < |B \cap X|$, there would be an element of $X \setminus A$ that is mapped to an element of X by such permutation, which is not possible for any element σ of $\text{Alt}(k) \times \text{Sym}(n - k)$.

By this claim, it is enough to verify the cases where the intersections with X have the same size, i.e., those with the same index.

We claim that X_1^+ and X_1^- are not in the same suborbit. To see this, assume that there exists $\sigma \in \text{Alt}(k) \times \text{Sym}(n - k)$ such that $\sigma(X_1^+) = X_1^-$. As $\sigma(X^+) = X^+$, we must have that $\sigma|_X$ is an even permutation and also that $\sigma(x_1) = x_1$ and $\sigma(k) = k$. Hence, $\sigma|_{\{1,2,\dots,k-1\}} = \sigma|_{X_1}$ is an even permutation, which makes $\sigma(X_1^+) = X_1^-$ impossible.

Consequently, the set $\{(X^+, A) : A \in S\}$ is a complete set of representatives of orbitals of $\text{Sym}(n)$ acting on the quasi k -subsets of $[n]$. It is straightforward to verify that the elements of this set yields the five lists of orbitals. This completes the proof. □

Using Theorem 6.2, we can explicitly define the schurian association scheme corresponding to the action of $\text{Sym}(n)$ on the cosets of the multiplicity-free subgroup $\text{Alt}(k) \times \text{Sym}(n - k)$. The association scheme corresponding to this action is the *quasi Johnson scheme*, denoted by $\mathcal{J}^{+\pm}(n, k)$, which is determined by the relations

$$\{\mathcal{O}_0^\pm, \mathcal{O}_1^\pm, \mathcal{O}_2, \mathcal{O}_3, \dots, \mathcal{O}_k\}.$$

6.4 The graphs in $\mathcal{J}^{+\pm}(n, k)$

For $2 \leq i \leq k$, we define the graph $J^\pm(n, k, i)$ to be the graph on $\binom{[n]}{k}^\pm$ whose edge set is \mathcal{O}_i . We will call the graph $\mathcal{K}^\pm(n, k) := J^\pm(n, k, k)$ the *quasi Kneser graph*.

The graph determined by \mathcal{O}_0^- is a perfect matching. We will denote the latter by $J^-(n, k, 0)$. The graphs determined by \mathcal{O}_1^+ and \mathcal{O}_1^- are called the *quasi Johnson graph of type +* and *quasi Johnson graph of type -*, respectively and we will denote them by $J^+(n, k, 1)$ and $J^-(n, k, 1)$, respectively.

In this subsection, we determine the structure of some graphs in the quasi Johnson scheme $\mathcal{J}^{\pm}(n, k)$ in terms of a graph product. We prove the next proposition.

Proposition 6.3. *For any $2 \leq i \leq k$, we have $J^{\pm}(n, k, i) = J(n, k, i) \bowtie K_2$.*

Proof. Let $i \in \{2, 3, \dots, k\}$, $V^+ = \{A^+ : A \in \binom{[n]}{k}\}$ and $V^- = \{A^- : A \in \binom{[n]}{k}\}$. Since $(A^+, B^+), (A^-, B^-) \in \mathcal{O}_i$ for any $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - i$, it is clear that the subgraph induced by V^+ and V^- are both isomorphic to $J(n, k, i)$. The edges of the form (A^+, B^-) are exactly those in $K_2 \times J(n, k, i)$. Hence, $J^{\pm}(n, k, i) = J(n, k, i) \bowtie K_2$. \square

Corollary 6.4. *Let $2 \leq i \leq k$. The eigenvalues of $J^{\pm}(n, k, i)$ are equal to 0 or of the form 2λ , where λ is an eigenvalue of $J(n, k, i)$.*

Proof. Let $A_{n,k,i}$ and $A_{n,k,i}^{\pm}$ be the adjacency matrices of the graphs $J(n, k, i)$ and $J^{\pm}(n, k, i)$, respectively. Since $J^{\pm}(n, k, i) = J(n, k, i) \bowtie K_2$, the adjacency matrix of $J^{\pm}(n, k, i)$ is

$$A_{n,k,i}^{\pm} = \begin{bmatrix} A_{n,k,i} & A_{n,k,i} \\ A_{n,k,i} & A_{n,k,i} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \otimes A_{n,k,i}.$$

Since the spectrum of 2×2 all-ones matrix is $\{0, 2\}$, the result follows. \square

Corollary 6.5. *The eigenvalues of the quasi Kneser graph $\mathcal{K}^{\pm}(n, k)$ are 0 or*

$$2(-1)^j \binom{n - k - j}{k - j},$$

for $0 \leq j \leq k$.

The graphs $J^+(n, k, 1)$ and $J^-(n, k, 1)$ are respectively two copies of the Johnson graph $J(n, k, 1)$, and the bipartite double cover of $J(n, k, 1)$. Consequently, the eigenvalues of $J^+(n, k, 1)$ are the eigenvalues of $J(n, k, 1)$, and the multiplicities are twice of that of $J(n, k, 1)$. The eigenvalues of $J^-(n, k, 1)$ are all $\pm\theta$, where θ is an eigenvalue of $J(n, k, 1)$.

6.5 The $[n - 1, 1]$ -module

Let $K = \text{Sym}([n - 1, 1])$ and let $\Omega = \binom{[n]}{k}^{\pm}$. The group K has two orbits on the quasi k -subsets of $[n]$. These two orbits of K are

$$\mathcal{S} = \left\{ A^{\pm} : A \in \binom{[n]}{k} \text{ and } n \in A \right\} \text{ and } \Omega \setminus \mathcal{S}.$$

Proposition 6.6. *The family \mathcal{S} is a maximum coclique of $\mathcal{K}^\pm(n, k)$.*

Proof. For any $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = 1$, we know that $(A^\pm, B^\pm) \notin \mathcal{O}_k$. Therefore, \mathcal{S} is a coclique of $\mathcal{K}^\pm(n, k)$.

To prove that \mathcal{S} is a maximum coclique, we use the Ratio Bound. By Corollary 6.5, the largest and the least eigenvalues of $\mathcal{K}^\pm(n, k)$ are respectively

$$2 \binom{n-k}{k} \text{ and } -2 \binom{n-k-1}{k-1}.$$

By the Ratio Bound, we have

$$\alpha(\mathcal{K}^\pm(n, k)) \leq \frac{2 \binom{n}{k}}{1 - \frac{2 \binom{n-k}{k}}{-2 \binom{n-k-1}{k-1}}} = \frac{2 \binom{n}{k}}{1 + \frac{n-k}{k}} = 2 \binom{n-1}{k-1} = |\mathcal{S}|.$$

□

By Theorem 3.7, we deduce the following corollary.

Corollary 6.7. *The eigenvalue $-2 \binom{n-k-1}{k-1}$ is afforded by the $[n-1, 1]$ -module.*

Therefore, the answer to Question 1.1 is affirmative for association scheme corresponding to the Gelfand pair (G, H) , where $G = \text{Sym}(n)$ and $H = \text{Alt}(k) \times \text{Sym}(n-k)$.

In the next theorem, we find the maximum cocliques of $\mathcal{K}^\pm(n, k)$.

Theorem 6.8. *The maximum cocliques of $\mathcal{K}^\pm(n, k)$ are of the form*

$$\left\{ A^\pm : A \in \binom{[n]}{k} \text{ and } x \in A \right\},$$

for some $x \in [n]$.

Proof. Let \mathcal{S} be a maximum coclique of $\mathcal{K}^\pm(n, k)$ and define

$$\mathcal{S}_+ = \mathcal{S} \cap \left\{ A^+ : A \in \binom{[n]}{k} \right\} \text{ and } \mathcal{S}_- = \mathcal{S} \cap \left\{ A^- : A \in \binom{[n]}{k} \right\}.$$

Note that $|\mathcal{S}| = 2 \binom{n-1}{k-1}$. Moreover, since $\mathcal{K}^\pm(n, k) = K(n, k) \bowtie K_2$ and the maximum cocliques of $K(n, k)$ have size $\binom{n-1}{k-1}$, we have $|\mathcal{S}_+| = |\mathcal{S}_-| = \binom{n-1}{k-1}$. That is, \mathcal{S}_- and \mathcal{S}_+ are maximum cocliques in the two copies of $K(n, k)$ in $\mathcal{K}^\pm(n, k)$. Therefore, there exists $a, b \in [n]$ such that

$$\mathcal{S}_+ = \left\{ A^+ : A \in \binom{[n]}{k} \text{ and } a \in A \right\} \text{ and } \mathcal{S}_- = \left\{ A^- : A \in \binom{[n]}{k} \text{ and } b \in A \right\}.$$

It is clear that $\mathcal{S} = \mathcal{S}_- \cup \mathcal{S}_+$ is a coclique of $\mathcal{K}^\pm(n, k)$ if and only if $a = b$. This completes the proof. □

7 The orbital scheme from line 4 of Table 1

Consider the Gelfand pair $(G, H) = (\text{Sym}(n), \text{Alt}(k) \times \text{Alt}(n - k))$. First, we will determine certain combinatorial objects that correspond to G/H . Then, we will prove the main result.

By [9], we first recall that

$$\mathbf{1}_H^G = \sum_{i=0}^k \left(\chi^{[n-i, i]} + \chi^{[2^i, 1^{n-2i}]} \right) + \chi^{[n-k, 1^k]} + \chi^{[k, 1^{n-k}]} + \chi^{[n-k+1, 1^{k-1}]} + \chi^{[k+1, 1^{n-k-1}]}.$$

Hence, $[n - 1, 1]$ is the second largest in dominance ordering in $\Lambda(n, H)$.

7.1 Combinatorial objects

For any k -subset A , let \underline{A} be the complement of A in $[n]$. Consider the set

$$\Omega_{n,k} = \left\{ (A^\pm, \underline{A}^\pm) : A \in \binom{[n]}{k} \right\}.$$

Note that $|\Omega| = 4 \binom{n}{k}$. We claim that the action of G on G/H is permutation equivalent to a certain action of $\text{Sym}(n)$ on $\Omega_{n,k}$. For any $\sigma \in \text{Sym}(n)$ and $(A^{*1}, \underline{A}^{*2}) \in \Omega_{n,k}$, define

$$\sigma((A^{*1}, \underline{A}^{*2})) := (\sigma(A^{*1}), \sigma(\underline{A}^{*2})).$$

Note that this is an induced action from the action given in (6.1). This gives an action of $\text{Sym}(n)$ on $\Omega_{n,k}$. Let us now prove that the stabilizer of an element of $\Omega_{n,k}$ is conjugate to H . Consider the element $(A^+, \underline{A}^+) \in \Omega_{n,k}$, where $A = \{1, 2, \dots, k\}$. If $\sigma \in \text{Sym}(n)$ fixes (A^+, \underline{A}^+) , then we have

$$\sigma(A^+) = A^+ \text{ and } \sigma(\underline{A}^+) = \underline{A}^+.$$

In other words, $\sigma|_A$ and $\sigma|_{\underline{A}}$ must be even permutations. Consequently, the stabilizer of (A^+, \underline{A}^+) is equal to H . One can also prove that H is also the stabilizer of (A^-, \underline{A}^-) , (A^+, \underline{A}^-) , and (A^-, \underline{A}^+) .

We omit the proof of the following proposition since it is straightforward.

Proposition 7.1. *The action of G on G/H is permutation isomorphic to the action of G on $\Omega_{n,k}$.*

7.2 The orbital scheme $\mathcal{J}^{++}(n, k)$

In this subsection, we determine the graphs in the orbital scheme corresponding to (G, H) .

Theorem 7.2. *An orbital of $G = \text{Sym}(n)$ in its action on $\Omega_{n,k}$ is one of the following.*

- 1) $\mathcal{O}_0^{+,+} := G. ((A^+, \underline{A}^+), (A^+, \underline{A}^+))$, where $A \in \binom{[n]}{k}$, which is the diagonal orbital.
- 2) $\mathcal{O}_0^{-,-} := G. ((A^+, \underline{A}^+), (A^-, \underline{A}^-))$, where $A \in \binom{[n]}{k}$.
- 3) $\mathcal{O}_0^{+,-} := G. ((A^+, \underline{A}^+), (A^+, \underline{A}^-))$, where $A \in \binom{[n]}{k}$.
- 4) $\mathcal{O}_0^{-,+} := G. ((A^+, \underline{A}^+), (A^-, \underline{A}^+))$, where $A \in \binom{[n]}{k}$.
- 5) $\mathcal{O}_1^{+,+} := G. ((A^+, \underline{A}^+), (B^+, \underline{B}^+))$, where $A, B \in \binom{[n]}{k}$ and $|A \cap B| = k - 1$.
- 6) $\mathcal{O}_1^{-,-} := G. ((A^+, \underline{A}^+), (B^-, \underline{B}^-))$, where $A, B \in \binom{[n]}{k}$ and $|A \cap B| = k - 1$.
- 7) $\mathcal{O}_1^{+,-} := G. ((A^+, \underline{A}^+), (B^+, \underline{B}^-))$, where $A, B \in \binom{[n]}{k}$ and $|A \cap B| = k - 1$.
- 8) $\mathcal{O}_1^{-,+} := G. ((A^+, \underline{A}^+), (B^-, \underline{B}^+))$, where $A, B \in \binom{[n]}{k}$ and $|A \cap B| = k - 1$.
- 9) $\mathcal{O}_i^+ := G. ((A^+, \underline{A}^+), (B^+, \underline{B}^+))$ for $2 \leq i \leq k$ and for $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - i$. In particular,

$$((X^+, \underline{X}^+), (Y^-, \underline{Y}^-)), ((X^-, \underline{X}^-), (Y^+, \underline{Y}^+)) \in \mathcal{O}_i^+$$

for any $X, Y \in \binom{[n]}{k}$ and $|X \cap Y| = k - i$.

- 10) $\mathcal{O}_i^- := G. ((A^+, \underline{A}^+), (B^+, \underline{B}^-))$ for $2 \leq i \leq k$ and $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - i$. In particular,

$$((X^+, \underline{X}^+), (Y^-, \underline{Y}^+)), ((X^-, \underline{X}^-), (Y^+, \underline{Y}^-)), ((X^-, \underline{X}^-), (Y^-, \underline{Y}^+)) \in \mathcal{O}_i^-$$

for any $X, Y \in \binom{[n]}{k}$ and $|X \cap Y| = k - i$.

Proof. Let $x_1 < x_2 < \dots < x_k$ be distinct elements of $[n] \setminus [k]$. Define

$$X_i = \{1, 2, \dots, k - i, x_1, x_2, \dots, x_i\}, \text{ for } 0 \leq i \leq k.$$

Note that $X_0 = \{1, 2, \dots, k\}$ and $X_k = \{x_1, x_2, \dots, x_k\}$. It is enough to prove that the set

$$S = \{(X_i^\pm, \underline{X}_i^\pm) : 0 \leq i \leq 1\} \cup \{(X_i^+, \underline{X}_i^+) : 2 \leq i \leq k\} \cup \{(X_i^+, \underline{X}_i^-) : 2 \leq i \leq k\},$$

consists of elements from distinct suborbits of $(X_0^+, \underline{X}_0^+)$. Recall that the stabilizer of $(X_0^+, \underline{X}_0^+)$ is $\text{Alt}(k) \times \text{Alt}(n - k)$. Similar to what we saw in the previous section, it is clear that an element of $\text{Alt}(k) \times \text{Alt}(n - k)$ cannot map $(X_i^\pm, \underline{X}_i^\pm)$ to $(X_j^\pm, \underline{X}_j^\pm)$, unless $i = j$.

Since $\text{Alt}(k) \times \text{Alt}(n - k)$ stabilizes $(X_0^+, \underline{X}_0^+)$ and $(X_0^-, \underline{X}_0^-)$, these two cannot be in the same suborbit. If $\sigma \in \text{Sym}(n)$ such that $\sigma(X_0^+, \underline{X}_0^-) = (X_0^-, \underline{X}_0^+)$, then it is straightforward that $\sigma \notin \text{Alt}(k) \times \text{Alt}(n - k)$. Hence, the four elements $(X_0^\pm, \underline{X}_0^\pm)$ are in different suborbits.

Assume now that there exists $\sigma \in \text{Alt}(k) \times \text{Alt}(n - k)$ such that $(X_1^+, \underline{X}_1^+)$ is mapped to $(X_1^-, \underline{X}_1^-)$ by σ . It is necessary that the set X_1 is fixed by σ setwise. It is clear that $\sigma(k) = k$ and $\sigma(x_1) = x_1$ since σ also fixes $X_0 = \{1, 2, \dots, k\}$ and \underline{X}_0 . We conclude that

$$\sigma|_{X_1} = \sigma|_{X_0 \setminus \{k\}}$$
 is an even permutation.

Consequently, σ cannot map X_0^+ to X_0^- . The other cases can be proved in the same way. Consequently, the elements $(X_1^\pm, \underline{X}_1^\pm)$ are in different suborbits of $\text{Alt}(k) \times \text{Sym}(n - k)$.

Finally, we assume that $2 \leq i \leq k$. We prove that $(X_i^+, \underline{X}_i^+)$ and $(X_i^+, \underline{X}_i^-)$ are not in the same suborbits by contradiction. Assume that $\sigma \in \text{Alt}(k) \times \text{Alt}(n - k)$ maps $(X_i^+, \underline{X}_i^+)$ to $(X_i^+, \underline{X}_i^-)$. Again, we note that σ fixes X_i and \underline{X}_i setwise. Combining this with the fact that σ permutes the elements of X_0 and $\sigma|_{X_i}$ is an even permutation, we deduce that

$$\sigma|_{\{1, 2, \dots, k-i\}} \text{ and } \sigma|_{\{x_1, x_2, \dots, x_i\}}$$

must be both even or both odd permutations. Without loss of generality, assume that they are both even permutations. Noting that $\sigma|_{\underline{X}_i}$ is an odd permutation (it maps \underline{X}_i^+ to \underline{X}_i^-), by a similar argument as above, we deduce that one of

$$\sigma|_{\{k-i+1, \dots, k\}} \text{ and } \sigma|_{\underline{X}_0 \setminus \{x_1, x_2, \dots, x_i\}}$$

must be even and the other one is an odd permutation. As $\sigma|_{\{1, 2, \dots, k-i\}}$ is an even permutation and $\sigma(X_0^+) = X_0^+$, we conclude that $\sigma|_{\{k-i+1, \dots, k\}}$ must be an even permutation and $\sigma|_{\underline{X}_0 \setminus \{x_1, x_2, \dots, x_i\}}$ is an odd permutation. In summary, $\sigma|_{\{x_1, x_2, \dots, x_i\}}$ is an even permutation and $\sigma|_{\underline{X}_0 \setminus \{x_1, x_2, \dots, x_i\}}$ is an odd permutation. This implies that $\sigma|_{\underline{X}_0}$ is an odd permutation. The latter is impossible. Therefore, the two elements that we started with cannot be in the same orbital.

Since the corresponding action of G has rank $2k + 6$, the set S consists of an element from each suborbits. It is easy to check that each element of S is a representative of the ten sets in the statement of the theorem. The rest of the proof follows by making the symmetric group $\text{Sym}(n)$ act on the pairs $((X_0^+, \underline{X}_0^+), T)$, where $T \in S$. □

We will denote the association scheme given by the orbitals in the previous theorem by $\mathcal{J}^{++}(n, k)$. It is not hard to check that all orbitals listed above are self-paired. Let $\mathcal{K}(n, k)$ be the graph corresponding to the orbital \mathcal{O}_k^+ .

Proposition 7.3. *The graph $\mathcal{K}(n, k)$ is isomorphic to two disjoint copies of the graph $K(n, k) \bowtie K_2$.*

Proof. Partition the vertices of $\mathcal{K}(n, k)$ into $\{V^{+,+}, V^{+,-}, V^{-,+}, V^{-,-}\}$, where

$$\begin{aligned} V^{+,+} &= \left\{ (A^+, \underline{A}^+) : A \in \binom{[n]}{k} \right\}, \\ V^{+,-} &= \left\{ (A^+, \underline{A}^-) : A \in \binom{[n]}{k} \right\}, \\ V^{-,+} &= \left\{ (A^-, \underline{A}^+) : A \in \binom{[n]}{k} \right\}, \\ V^{-,-} &= \left\{ (A^-, \underline{A}^-) : A \in \binom{[n]}{k} \right\}. \end{aligned}$$

By definition of \mathcal{O}_k^+ , it is clear that there is no edge between a vertex in $V^{+,+} \cup V^{-,-}$ and a vertex in $V^{+,-} \cup V^{-,+}$. In other words, $\mathcal{K}(n, k)$ is disconnected. A vertex (A^+, \underline{A}^+) is adjacent to all vertices of the form (B^+, \underline{B}^+) and (B^-, \underline{B}^-) , for all k -subsets B with the property that $A \cap B = \emptyset$. This proves that the subgraph of $\mathcal{K}(n, k)$ induced by $V^{+,+} \cup V^{-,-}$ is in fact isomorphic to $K(n, k) \bowtie K_2$. One can also prove with the same argument that the subgraph induced by $V^{-,+} \cup V^{+,-}$ is also isomorphic to $K(n, k) \bowtie K_2$. \square

7.3 The $[n - 1, 1]$ -module

By Lemma 3.3, $K = \text{Sym}([n - 1, 1])$ has two orbits on $\Omega_{n,k}$. It is immediate that the partition from these orbits is $\pi = \{\mathcal{S}, \Omega \setminus \mathcal{S}\}$, where

$$\mathcal{S} = \left\{ (A^\pm, \underline{A}^\pm) : A \in \binom{[n]}{k} \text{ and } n \in A \right\}.$$

It is clear that \mathcal{S} is a coclique of $\mathcal{K}(n, k)$ since for any $(A^\pm, \underline{A}^\pm), (B^\pm, \underline{B}^\pm) \in \mathcal{S}$, $|A \cap B| \geq 1$. We claim that \mathcal{S} is also a maximum coclique of $\mathcal{K}(n, k)$. To do this, we use the Ratio Bound as follows.

The proof of the following proposition is identical to the proof of Corollary 6.5.

Proposition 7.4. *The eigenvalues of $\mathcal{K}(n, k)$ are either 0 or*

$$2(-1)^j \binom{n - k - j}{k - j},$$

for $0 \leq j \leq k$.

From this proposition, we deduce that the smallest eigenvalue of the graph $\mathcal{K}(n, k)$ is $-2\binom{n-k-1}{k-1}$ and the largest eigenvalue is $2\binom{n-k}{k}$. By the Ratio Bound, we have

$$\alpha(\mathcal{K}(n, k)) \leq \frac{4\binom{n}{4}}{1 - \frac{2\binom{n-k}{k}}{-2\binom{n-k-1}{k-1}}} = 4\binom{n-1}{k-1}.$$

Consequently, \mathcal{S} is a coclique of maximum size. By Theorem 3.7, we deduce that the least eigenvalue $-2\binom{n-k-1}{k-1}$ is the eigenvalue afforded by the $[n - 1, 1]$ -module.

Remark 7.5. In this subsection, we proved that the $[n - 1, 1]$ -module is a subspace of the eigenspace corresponding to the eigenvalue $-2\binom{n-k-1}{k-1}$. However, this eigenspace is much larger than the $[n - 1, 1]$ -module since the graph is a disjoint union of two copies of the same graph. The other subspace of the eigenspace is the $[2, 1^{n-2}]$ -module since the eigenvalue afforded by the $[2, 1^{n-2}]$ -module is also equal to $-2\binom{n-k-1}{k-1}$. To prove this, consider the $(2k + 1)$ -cycle

$$x = (1, k + 1, 2, k + 2, \dots, i, k + i, i + 1, \dots, k, 2k, 2k + 1).$$

Note that if $A = \{1, 2, \dots, k\}$ and $B = \{k + 1, \dots, 2k\}$, then $x.A = B$. Since $|A \cap B| = 0$, there exists an edge between (A^+, \underline{A}^+) and (B^+, \underline{B}^+) in $\mathcal{K}(n, k)$. Moreover, since $x.A = B$ we can identify (A^+, \underline{A}^+) and (B^+, \underline{B}^+) respectively with the cosets $H = \text{Alt}(k) \times \text{Alt}(n - k)$ and xH . Clearly $xH \subset \text{Alt}(n)$ and (H, xH) corresponds to an edge in $\mathcal{K}(n, k)$. Now, note that the smallest eigenvalue is afforded by the $[n - 1, 1]$ -module so we have

$$\sum_{h \in H} \chi^{[n-1,1]}(xh) = \frac{|H|}{2\binom{n-k}{k}} \left(-2\binom{n-k-1}{k-1} \right).$$

Using this equality, the eigenvalue corresponding to $[2, 1^{n-2}]$ -module is

$$\frac{2\binom{n-k}{k}}{|H|} \sum_{h \in H} \chi^{[2,1^{n-2}]}(xh) = \frac{2\binom{n-k}{k}}{|H|} \sum_{h \in H} \chi^{[1^n]}(xh) \chi^{[n-1,1]}(xh) = -2\binom{n-k-1}{k-1}.$$

8 The orbital scheme from line 5 of Table 1

Let $G = \text{Sym}(n)$, $H = (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)$ and consider the Gelfand pair (G, H) . Similar to the previous cases, we will introduce certain combinatorial objects that correspond to G/H . Then, we will find an orbital graph in the orbital scheme that gives an affirmative answer to Question 1.1.

8.1 Combinatorial objects

Recall that $\mathcal{I}_{n,k}$ is the set of all injections from $[k]$ to $[n]$. Given $A \in \mathcal{I}_{n,k}$, let $\text{Im}(A)$ be the image of the map A and define

$$\underline{\mathcal{I}}_{n,k}(A) := \{g : [n - k] \rightarrow [n] \setminus \text{Im}(A) : g \text{ is injective}\}.$$

Since we may view an element of $\mathcal{I}_{n,k}$ as a sequence of the form $A = (a_1, a_2, \dots, a_k)$ with distinct entries, we have an action of $\text{Sym}(k)$ on $\mathcal{I}_{n,k}$ through

$$\sigma(A) := (a_{\sigma(1)}, a_{\sigma(2)}, \dots, a_{\sigma(k)}),$$

for any $\sigma \in \text{Sym}(k)$. Similarly, the group $\text{Sym}(n - k)$ also acts on any element of $\mathcal{I}_{n,n-k}$ by permuting the indices. These two actions give an induced action of

$\text{Sym}(k) \times \text{Sym}(n - k)$ on $\{A\} \times \mathcal{I}_{n,k}(A)$, for any $A \in \mathcal{I}_{n,k}$. Now, define $\mathcal{T}_{n,k} := \{(A, B) : A \in \mathcal{I}_{n,k} \text{ and } B \in \underline{\mathcal{I}}_{n,k}(A)\}$ and consider the relation \mathcal{R} on $\mathcal{T}_{n,k}$ such that $(A, B)\mathcal{R}(C, D)$ if and only if there exists $\sigma \in (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)$ such that

$$\sigma(A) = C \text{ and } \sigma(B) = D.$$

Let us compute the number of equivalence classes of \mathcal{R} . For any $(A, B) \in \mathcal{T}_{n,k}$, the equivalence class of \mathcal{R} that contains (A, B) is

$$\overline{(A, B)} := \{(\sigma(A), \sigma(B)) : \sigma \in (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)\}$$

Let $S = \{a_1, a_2, \dots, a_k\}$ such that $a_1 < a_2 < \dots < a_k$ and $\underline{S} := [n] \setminus S = \{b_1, b_2, \dots, b_{n-k}\}$. Consider the set $T = \{(A, B) \in \mathcal{T}_{n,k} : \text{Im}(A) = S\}$. The group $(\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)$ acts intransitively on T since the orbit counting lemma gives

$$\begin{aligned} & \frac{2}{k!(n - k)!} \sum_{\sigma \in (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)} |\{(A, B) \in T : \sigma.A = A \text{ and } \sigma.B = B\}| \\ &= \frac{2}{k!(n - k)!} k!(n - k)! = 2. \end{aligned}$$

Therefore, the set S determines two orbits which are

$$\begin{aligned} (S, \underline{S})^+ &:= \overline{((a_1, a_2, \dots, a_k), (b_1, b_2, \dots, b_{n-k}))}, \\ (S, \underline{S})^- &:= \overline{((a_2, a_1, \dots, a_k), (b_1, b_2, \dots, b_{n-k}))}. \end{aligned}$$

Note that $(S, \underline{S})^+$ and $(S, \underline{S})^-$ are both fixed by $(\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)$ and they are swapped by any odd permutation in $(\text{Sym}(k) \times \text{Sym}(n - k))$.

For the remainder of this section, we let

$$\Omega_{n,k} := \left\{ (S, \underline{S})^\pm : S \in \binom{[n]}{k} \right\}.$$

8.2 Action of the symmetric group

For any $A = \{a_1, a_2, \dots, a_k\}$ such that $a_1 < a_2 < \dots < a_k$, and $\sigma \in \text{Sym}(n)$, let $\rho(\sigma, A) \in \text{Sym}(k) \times \text{Sym}(n - k)$ be the unique permutation that acts on the indices $\{1, 2, \dots, k\} \times \{k + 1, \dots, n\}$ by reordering the entries of $\sigma(A)$ and $\sigma(\underline{A})$ in an increasing order. To see this in more details, assume $\underline{A} = \{b_{k+1}, b_{k+2}, \dots, b_n\}$ such that $b_{k+1} < b_{k+2} < \dots < b_n$. If

$$\sigma(a_{i_1}) < \sigma(a_{i_2}) < \dots < \sigma(a_{i_k}) \text{ and } \sigma(b_{j_{k+1}}) < \sigma(b_{j_{k+2}}) < \dots < \sigma(b_{j_n}),$$

then $\rho = \rho(\sigma, A) \in \text{Sym}(k) \times \text{Sym}(n - k)$ such that

$$\rho(\ell) = i_\ell \text{ for } \ell \in \{1, 2, \dots, k\} \text{ and } \rho(t) = j_t \text{ for } t \in \{k + 1, k + 2, \dots, n\}.$$

Recall that $H = (\text{Alt}(k) \times \text{Alt}(n - k)) \cap \text{Alt}(n)$. For any $\sigma \in \text{Sym}(n)$ and $S \in \binom{[n]}{k}$, define

$$\sigma((S, \underline{S})^*) := \begin{cases} (\sigma(S), \sigma(\underline{S}))^+ & \text{if } \rho(\sigma, S) \in H \text{ and } * = + \\ (\sigma(S), \sigma(\underline{S}))^- & \text{if } \rho(\sigma, S) \in H \text{ and } * = - \\ (\sigma(S), \sigma(\underline{S}))^+ & \text{if } \rho(\sigma, S) \notin H \text{ and } * = - \\ (\sigma(S), \sigma(\underline{S}))^- & \text{if } \rho(\sigma, S) \notin H \text{ and } * = +. \end{cases}$$

This yields a transitive action of $\text{Sym}(n)$ on $\Omega_{n,k}$. Moreover, if $S = \{1, 2, \dots, k\}$, then a permutation σ is in the stabilizer of $(S, \underline{S})^+$ if and only if $\sigma(S) = S$ and $\rho(\sigma, S) \in H$. Since the unique permutation that reorders the entries of $\sigma(S)$ and $\sigma(\underline{S})$ in an increasing order is σ^{-1} , we deduce that $\sigma = \rho(\sigma, S)^{-1}$. Consequently, σ fixes $(S, S)^+$ if and only if $\sigma \in H$. Hence,

$$\text{Stab}(\text{Sym}(n), (S, \underline{S})^+) = (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n).$$

Similarly,

$$\text{Stab}(\text{Sym}(n), (S, \underline{S})^-) = (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n).$$

The following proposition can be easily verified by using the fact that there is a unique conjugacy class of subgroups isomorphic to H in $\text{Sym}(n)$.

Proposition 8.1. *The action of $\text{Sym}(n)$ on the cosets of $(\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)$ is permutation equivalent to the action of $\text{Sym}(n)$ on $\Omega_{n,k}$.*

8.3 The orbital scheme $\mathcal{J}^\pm(n, k)$

We will denote the orbital scheme obtained from the Gelfand pair (G, H) , where $G = \text{Sym}(n)$, and $H = (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(n)$ by $\mathcal{J}^\pm(n, k)$. This association scheme is also known as the bipartite double of the Johnson scheme $\mathcal{J}(n, k)$.

Theorem 8.2. *An orbital of $G = \text{Sym}(n)$ acting on the $\Omega_{n,k}$ is one of the following.*

- 1) $\mathcal{O}_0^+ := G \cdot ((A, \underline{A})^+, (A, \underline{A})^+)$, where $A \in \binom{[n]}{k}$.
- 2) $\mathcal{O}_0^- := G \cdot ((A, \underline{A})^+, (A, \underline{A})^-)$, where $A \in \binom{[n]}{k}$.
- 3) $\mathcal{O}_i^- := G \cdot ((A, \underline{A})^+, (B, \underline{B})^-)$, for $1 \leq i \leq k$ and for $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - i$. In particular, \mathcal{O}_i^- contains $((X, \underline{X})^-, (Y, \underline{Y})^+)$ for any $X, Y \in \binom{[n]}{k}$ such that $|A \cap B| = k - i$.
- 4) $\mathcal{O}_i^+ := G \cdot ((A, \underline{A})^+, (B, \underline{B})^+)$, for $1 \leq i \leq k$ and for $A, B \in \binom{[n]}{k}$ such that $|A \cap B| = k - i$. In particular, \mathcal{O}_i^+ contains $((X, \underline{X})^-, (Y, \underline{Y})^-)$, for any $X, Y \in \binom{[n]}{k}$ such that $|X \cap Y| = k - i$.

Proof. We note first that the action of $\text{Sym}(n)$ on $\Omega_{n,k}$ has rank $2k + 2$ (see [9] for details). Consider the k distinct elements $x_1 < x_2 < \dots < x_k$ of $[n] \setminus \{1, 2, \dots, k\}$. Let $X_i = \{1, 2, \dots, k-i, x_1, x_2, \dots, x_i\}$ for $0 \leq i \leq k$. Note that $X_0 = \{1, 2, \dots, k\}$, $X_k = \{x_1, x_2, \dots, x_k\}$. We claim that the set

$$S = \{(X_i, \underline{X}_i)^+ : 0 \leq i \leq k\} \cup \{(X_i, \underline{X}_i)^- : 0 \leq i \leq k\}$$

has $2k + 2$ elements in different suborbits of $H = (\text{Sym}(k) \times \text{Sym}(n - k)) \cap \text{Alt}(k)$.

First, note that if $A \in \binom{[n]}{k}$ such that $|A \cap X| = i$, then $((X, \underline{X})^+, (A, \underline{A})^*)$ can only be mapped by H to an element of the form $((X, \underline{X})^+, (B, \underline{B})^\dagger)$, where $|B \cap X| = i$. Now, if $\sigma \in \text{Sym}(n)$ such that $\sigma((X_i, \underline{X}_i)^+) = (X_i, \underline{X}_i)^-$, then by definition we must have $\rho(\sigma, X_i) \notin H$. Hence, we can assume without loss of generality that $\sigma|_{X_i}$ is an even permutation and $\sigma|_{\underline{X}_i}$ is an odd permutation. Therefore, $\sigma|_{X_0 \cap X_i}$ and $\sigma|_{X_0 \cap \underline{X}_i}$ must have the same sign, whereas $\sigma|_{\underline{X}_0 \cap X_i}$ and $\sigma|_{\underline{X}_0 \cap \underline{X}_i}$ have opposite signs. Using this, one can easily prove that $\sigma((X_0, \underline{X}_0))^+$ cannot be equal to $(X_0, \underline{X}_0)^+$, which is a contradiction. This completes the proof. \square

Let $\mathcal{K}^+(n, k)$ and $\mathcal{K}^-(n, k)$ be the two orbital graphs corresponding to \mathcal{O}_k^+ and \mathcal{O}_k^- , respectively. By definition of the orbital \mathcal{O}_k^\pm , the following lemma follows immediately.

Lemma 8.3. *We have $\mathcal{K}^-(n, k) = K_2 \times K(n, k)$ and $\mathcal{K}^+(n, k)$ is a disjoint union of two copies of $K(n, k)$.*

Proof. In $\mathcal{K}^-(n, k)$, the edges are of the form $((A, \underline{A})^+, (B, \underline{B})^-)$ or $((A, \underline{A})^-, (B, \underline{B})^+)$, where $|A \cap B| = 0$. Since this graph is bipartite, we clearly have $\mathcal{K}^-(n, k) = K_2 \times K(n, k)$.

The graph $\mathcal{K}^+(n, k)$ is disconnected since there is no edge between vertices of the form $(A, \underline{A})^+$ and $(B, \underline{B})^-$, for any $A, B \in \binom{[n]}{k}$. The subgraph of $\mathcal{K}^-(n, k)$ induced by all vertices of the form $(A, \underline{A})^+$, for all $A \in \binom{[n]}{k}$, is isomorphic to $K(n, k)$. The same holds for the vertices of the form $(A, \underline{A})^-$, for all $A \in \binom{[n]}{k}$. This completes the proof. \square

8.4 The $[n - 1, 1]$ -module

Let $K = \text{Sym}([n - 1, 1])$. The orbit partition induced by K on $\Omega_{n,k}$ is $\pi = \{\mathcal{S}, \Omega_{n,k} \setminus \mathcal{S}\}$, where

$$\mathcal{S} = \left\{ (A, \underline{A})^\pm : A \in \binom{[n]}{k} \text{ and } n \in A \right\}.$$

We note that $|\mathcal{S}| = 2 \binom{n-1}{k-1}$. It is clear that \mathcal{S} is a coclique of both orbital graphs of $\mathcal{K}^-(n, k)$ and $\mathcal{K}^+(n, k)$, however, it is not a maximum coclique of the one corresponding to $\mathcal{K}^-(n, k)$ since the latter is bipartite. That is, the graph $\mathcal{K}^-(n, k)$ gives an affirmative answer to the first part of Question 1.1, however, the second part of the question is not satisfied.

Theorem 8.4. *The least eigenvalue of $\mathcal{K}^-(n, k)$ is afforded by the $[1^n]$ -module.*

Proof. By Theorem 3.7, the eigenvalue afforded by the $[n - 1, 1]$ -module is

$$-\frac{2\binom{n-k}{k}\binom{n-1}{k-1}}{2\binom{n}{k} - 2\binom{n-1}{k-1}}.$$

Since $\mathcal{K}^-(n, k)$ is bipartite, its smallest eigenvalue is equal to $-\binom{n-k}{k}$. One can verify that this eigenvalue is afforded by the $[1^n]$ -module by using Theorem 3.1. \square

Theorem 8.5. *The least eigenvalue of $\mathcal{K}^+(n, k)$ is afforded by the $[n - 1, 1]$ -module.*

Proof. Since \mathcal{S} is a coclique, by Theorem 3.7, the eigenvalue $-2\frac{\binom{n-k}{k}\binom{n-1}{k-1}}{2\binom{n-1}{k}}$ is the eigenvalue afforded by the $[n - 1, 1]$ -module. As $\mathcal{K}^+(n, k)$ is a disjoint union of two copies of $K(n, k)$, its smallest eigenvalue is $-\binom{n-k-1}{k-1} = -\frac{\binom{n-k}{k}\binom{n-1}{k-1}}{\binom{n-1}{k}}$. \square

9 The orbital scheme from line 6 of Table 1

In this section, we prove that the answer to Question 1.1 is affirmative for the group in line 6 of Table 1. First, we prove that Question 1.1 is affirmative for the Gelfand pair $(\text{Sym}(2k), \text{Sym}(2) \wr \text{Sym}(k))$. Then, we use a graph isomorphism to prove that Question 1.1 also holds for the Gelfand pair $(\text{Sym}(2k + 1), \text{Sym}(2) \wr \text{Sym}(k))$.

9.1 Question 1.1 for the Gelfand pair $(\text{Sym}(2k), \text{Sym}(2) \wr \text{Sym}(k))$

For any $k \geq 2$, the action of $\text{Sym}(2k)$ on the cosets of $\text{Sym}(2) \wr \text{Sym}(k)$ is multiplicity-free (see [9]). We note that the second largest in dominance ordering in $\Lambda(2k, \text{Sym}(2) \wr \text{Sym}(k))$ is $[2k - 2, 2]$. The action of $\text{Sym}(2k)$ on the cosets of $\text{Sym}(2) \wr \text{Sym}(k)$ is equivalent to the action of $\text{Sym}(2k)$ on the perfect matchings of the complete graph K_{2k} . A perfect matching of K_{2k} is a partition of the set $[2k]$ into k subsets of size 2. We will denote the set of all perfect matchings of K_{2k} by \mathcal{P}_k .

For any $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_t]$, let $\mathcal{O}_{[2\lambda_1, 2\lambda_2, \dots, 2\lambda_t]}$ be the set of all pairs (P, Q) of elements of \mathcal{P}_k such that $P \cup Q$ is a union of t cycles of length $2\lambda_1, 2\lambda_2, \dots, 2\lambda_t$. We note that an edge is considered a 2-cycle in this definition.

For any $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_k] \vdash k$, we define $2\lambda := [2\lambda_1, 2\lambda_2, \dots, 2\lambda_k]$. The association scheme arising from $(\text{Sym}(2k), \text{Sym}(2) \wr \text{Sym}(k))$ is the well-known *perfect matching association scheme*. The orbitals of the corresponding group action are all $\mathcal{O}_{2\lambda}$, where $\lambda \vdash k$. Next, we prove that the answer to Question 1.1 follows from a result in [10, 15].

For any n , we let

$$n!! = \begin{cases} n \times (n - 2) \times \dots \times 3 \times 1 & \text{if } n \text{ is odd} \\ n \times (n - 2) \times \dots \times 4 \times 2 & \text{otherwise.} \end{cases}$$

Using this notation, it is not hard to see that

$$|\mathcal{P}_k| = (2k - 1)!!.$$

For any $\mathcal{F} \subset \mathcal{P}_k$, we say that \mathcal{F} is intersecting if $|P \cap Q| \geq 1$, for any P and Q in \mathcal{F} . We state the following theorem about intersecting families of \mathcal{P}_k .

Theorem 9.1 ([10, 15]). *If $\mathcal{F} \subset \mathcal{P}_k$ is intersecting, then $|\mathcal{F}| \leq (2k - 3)!!$. Moreover, equality holds if and only if \mathcal{F} consists of all perfect matchings with a fixed edge, i.e., an orbit of size $(2k - 3)!!$ of a conjugate of $\text{Sym}(2k - 2) \times \text{Sym}(2)$.*

To prove Theorem 9.1, the authors relied on cocliques in the so-called *perfect matching derangement graph* $P(k)$. The vertices of this graph consist of the elements of \mathcal{P}_k and two perfect matchings are adjacent if they are not intersecting. The graph $P(k)$ is the union of all orbital graphs corresponding to $\mathcal{O}_{2\lambda}$, where $\lambda \vdash k$ and 2λ does not contain any part of size 2. It follows from Theorem 9.1 that any orbital graph which is a subgraph of $P(k)$ gives an affirmative answer to Question 1.1 (a). Using the upper bound on the dimension of the Specht modules in $\Lambda(2k, \text{Sym}(2) \wr \text{Sym}(k))$ given in [3, Lemma 3.7] (or [2, Lemma 3.2]) and the “trace trick” as given in [10, Theorem 7.2], one can prove that the smallest eigenvalue of perfect matching derangement graph $P(k)$ is equal to $-(2k - 2)!!$, and the latter is afforded by the $[2k - 2, 2]$ -module.

However, when considering the subgraphs of $P(k)$ corresponding to orbital graphs in the association scheme, the trace trick fails. In [15], Lindzey showed that in the orbital graph $\mathcal{O}_{[2k]}$, the least eigenvalue is afforded by $[2k - 2, 2]$, and is equal to $-(2k - 2)!!$. Thus Question 1.1 (b) is affirmative.

9.2 Question 1.1 for the Gelfand pair $(\text{Sym}(2k + 1), \text{Sym}(2) \wr \text{Sym}(k))$

Let k be a positive integer. For any $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_t] \vdash k$, under the assumption that $\lambda_0 = k + 1$, we let

$$\mathcal{I}(\lambda) := \{i \in \{1, 2, \dots, t\} \mid \lambda_{i-1} > \lambda_i\}.$$

For any $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_t] \vdash k$, define the partitions of $2k + 1$ given by

$$\begin{cases} 2\lambda^{(i)} + 1 & := [2\lambda_1, 2\lambda_2, \dots, 2\lambda_i + 1, \dots, 2\lambda_t], \text{ for } i \in \mathcal{I}(\lambda) \\ 2\lambda^{(t+1)} + 1 & := [2\lambda_1, 2\lambda_2, \dots, 2\lambda_t, 1]. \end{cases}$$

Further, for any $\lambda = [\lambda_1, \lambda_2, \dots, \lambda_t] \vdash k$ we define the set

$$2\lambda + 1 = \{2\lambda^{(i)} + 1 : i \in \mathcal{I}(\lambda) \cup \{t + 1\}\}.$$

Finally, define the set

$$\Lambda_k := \bigcup_{\lambda \vdash k} 2\lambda + 1.$$

The group $\text{Sym}(2) \wr \text{Sym}(k)$ is a multiplicity-free subgroup of $\text{Sym}(2k + 1)$ [9]. Using [9], we know that

$$\mathbf{1}_{\text{Sym}(2) \wr \text{Sym}(k)}^{\text{Sym}(2k+1)} = \sum_{\mu \in \Lambda_k} \chi^\mu.$$

The action of $\text{Sym}(2k + 1)$ on cosets of $\text{Sym}(2) \wr \text{Sym}(k)$ is equivalent to its action on the set \mathcal{Q}_k of all partitions of $[2k + 1]$ into k sets of size two and a singleton. Such a partition in \mathcal{Q}_k is called a *quasi-perfect matching* of K_{2k+1} . It is clear that

$$|\mathcal{Q}_k| = (2k + 1)!!.$$

In the next lemma, we determine the orbitals of $\text{Sym}(2k + 1)$ in its action on quasi-perfect matchings of K_{2k+1} . We omit the proof of this lemma since it is similar to how the orbital graphs of the action of $\text{Sym}(2k)$ on cosets of $\text{Sym}(2) \wr \text{Sym}(k)$ are obtained (see [11] for details).

Lemma 9.2. *An orbital of $\text{Sym}(2k + 1)$ in its action on quasi-perfect matchings of K_{2k+1} is of the form:*

- (i) $\mathcal{O}_{[2\lambda_1, \dots, 2\lambda_{i+1}, \dots, 2\lambda_t]}$, for some partition $[\lambda_1, \lambda_2, \dots, \lambda_t] \vdash k$, where $(P, P') \in \mathcal{O}_{[2\lambda_1, \dots, 2\lambda_{i+1}, \dots, 2\lambda_t]}$ if and only if $P \cup P'$ is a disjoint union of $t - 1$ cycles of length $2\lambda_1, \dots, 2\lambda_{i-1}, 2\lambda_{i+1}, \dots, 2\lambda_t$ and a path of length $2\lambda_i + 1$.
- (ii) $\mathcal{O}_{[2\lambda_1, \dots, 2\lambda_t, 1]}$, for some partition $[\lambda_1, \dots, \lambda_t] \vdash k$, where $(P, P') \in \mathcal{O}_{[2\lambda_1, \dots, 2\lambda_t, 1]}$ if and only if $P \cup P'$ is a disjoint union of $t - 1$ cycles of length $2\lambda_1, \dots, 2\lambda_t$ and an isolated vertex.

For any $k \geq 2$, let $Q(k)$ be the graph consisting of the union of all orbital graphs corresponding to \mathcal{O}_μ , such that $\mu \in \Lambda_k$ does not contain any part of size 1 or 2. We prove the next result about the relation between $Q(k)$ and the perfect matching derangement graph.

Lemma 9.3. *For any $k \geq 2$, there exists graph isomorphism φ from $Q(k)$ to $P(k+1)$.*

Proof. Consider the map $\varphi : \mathcal{Q}_k \rightarrow \mathcal{P}_{k+1}$ such that any $Q \in \mathcal{Q}_k$ is mapped to the element $\varphi(Q)$ of \mathcal{P}_{k+1} obtained from Q by replacing the singleton $\{a\} \in Q$ by $\{a, 2k + 2\}$. It is clear $\varphi(Q) \in \mathcal{P}_{k+1}$ and φ is well defined. By uniqueness of the singleton in any element \mathcal{Q}_k , it is clear that φ is injective. The surjectivity is obtained by replacing the 2-subset $\{a, 2k + 2\}$ in any element of \mathcal{P}_{k+1} by $\{a\}$. Hence, φ is a bijection.

It remains to prove that φ preserves adjacency and non-adjacency. For any $Q, Q' \in \mathcal{Q}_k$ such that $Q \sim_{Q(k)} Q'$, it is clear that $\varphi(Q) \cup \varphi(Q')$ does not contain any isolated edge, otherwise, $Q \cup Q'$ would contain a 2-cycle or an isolated vertex. If $P = \varphi(Q), P' = \varphi(Q') \in P(k + 1)$ are adjacent, then $P \cup P'$ does not contain a 2-cycle, so the removal of the vertex $2k + 2$ cannot give rise to a graph with an isolated vertex or a graph with 2-cycle. Consequently, φ is an isomorphism. \square

It is not hard to see that the isomorphism φ also induces an isomorphism on the orbital graphs that are subgraphs of $Q(k)$ and $P(k+1)$.

An immediate corollary of the above lemma and Theorem 9.1 is the following.

Theorem 9.4. *If $\mathcal{F} \subset \mathcal{Q}_k$ is intersecting, then $|\mathcal{F}| \leq (2k-1)!!$. Equality holds if and only if \mathcal{F} is an orbit of size $(2k-1)!!$ of a conjugate of $\text{Sym}([2k, 1])$ or a conjugate of $\text{Sym}([2k-1, 2])$.*

We note that the partitions $[2k, 1]$ and $[2k-1, 2]$ are exactly the two resulting partitions obtained by applying the Branching Rule from $\text{Sym}(2k+2)$ to $\text{Sym}(2k+1)$ (see [17] for details) on the partition $[2k, 2]$, which is the one that determines the smallest eigenvalue of $P(k+1)$.

Since $Q(k)$ is a union of orbital graphs of $\text{Sym}(2k+1)$ acting on \mathcal{Q}_k , we conclude that the orbit of $\text{Sym}([2k, 1])$ of size $(2k-1)!!$ is a coclique of any orbital graph contained in $Q(k)$. Hence, Question 1.1 (a) is affirmative.

For Question 1.1 (b), we consider the orbital graph X corresponding to $\mathcal{O}_{[2k+1]}$. Due to Question 1.1 (a) being affirmative, $-(2k-2)!!$ is the eigenvalue of X afforded by $[2k, 1]$ -module. Let Y be the subgraph of $P(k+1)$ which is the orbital graph of $\text{Sym}(2k+2)$ acting on \mathcal{P}_{k+1} corresponding to the partition $[2k+2]$. It is known that the smallest eigenvalue $-(2k-2)!!$ of Y is afforded by the $[2k, 2]$ -module and is the least eigenvalue of Y . Therefore, $-(2k-2)!!$ is also the smallest eigenvalue of X , since X and Y are isomorphic.

10 Conclusion and future work

In this paper, we proved in Theorem 1.6 that for any Gelfand pair (G, H) , where $G = \text{Sym}(n)$ and H is listed in lines 1-6 of Table 1, there exists a graph in the corresponding orbital scheme for which Question 1.1 is affirmative. We also gave an example where the answer to Question 1.1 is negative.

We do not know if Question 1.1 is true in general for all multiplicity-free subgroups H of $\text{Sym}(n)$ such that the second largest in dominance ordering in $\Lambda(n, H)$ is $[n-1, 1]$. Therefore, we pose the following problem.

Problem 10.1. Determine whether Question 1.1 is true for all other Gelfand pairs $(\text{Sym}(n), H)$ in which $[n-1, 1]$ is the second largest in $\Lambda(n, H)$.

For orbital schemes arising from other multiplicity-free subgroups, we expect the case where the second largest in dominance ordering in $\Lambda(n, H)$ is not equal to $[n-1, 1]$ to be more complicated than the case considered in this paper. We provide some computational results on the small multiplicity-free subgroups obtained from Sagemath [18] in Table 2.

We also ask the following problem.

Problem 10.2. Find an EKR type theorem for partially 2-intersecting families of U_k . That is, determine the maximum cocliques of the folded Johnson graph $\overline{J}(2k, k)$.

Group	n	index	rank	Second largest	Question 1.1(a)	Question 1.1(b)
$AGL(1, 5) \cap A_5$	5	12	4	[3, 2]	Yes	Yes
$AGL(1, 5)$	5	6	2	[2 ² , 1]	No	No
$PSL(2, 5)$	6	12	4	[3 ²]	Yes	Yes
$PGL(2, 5)$	6	6	2	[2 ³]	No	No
$AGL(1, 7)$	7	120	7	[4, 3]	No	No
$PSL(3, 2)$	7	30	4	[4, 3]	Yes	No
$AGL(1, 8)$	8	240	8	[5, 1 ³]	No	No
$PGL(2, 7)$	8	120	5	[4, 4]	No	No
$AGL(3, 2)$	8	30	4	[4 ²]	Yes	No
$Sym(2) \wr Alt(3)$	6	30	5	[4, 2]	Yes	No
$Sym(2) \wr Alt(4)$	6	30	5	[6, 2]	Yes	No

Table 2: Answer to Question 1.1 for small multiplicity-free subgroup.

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