ON THE CONSTRUCTION OF ARCS USING QUADRICS

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Dedicated to the memory of Alan Rahilly, 1947 – 1992

Abstract

A new method of constructing arcs in projective space is given. It is a generalisation of the fact that a normal rational curve can be given by the complete intersection of a set of quadrics. The non-classical 10-arc of $PG(4,9)$ together with its special point is the set of derived points of a cubic primal. This property is shared with the normal rational curve of this space.

1. INTRODUCTION AND NOTATION

This paper shows that *quadrics* (quadratic primals or hypersurfaces in finite projective space) are related to k-arcs and their associated curves in a fundamental way. The methods of classical algebraic geometry are necessary for much of the discussion; see [**1**,**26**]. Before delving into these connections we shall give the reader a gentle introduction to the known constructions and theory of arcs.

The geometries which are the object for this discussion are, in the main, the finite projective spaces $PG(n,q)$ of dimension n over $GF(q)$, for $n \geq 2$, and $q = p^h$, p prime. However, many of the results are valid for any projective space over a general field F. Here is some notation: a subspace of dimension m is often denoted $[m]$. A space of dimension m, every point of which is associated with a quadric of $[n]$, is denoted $Q_n[m]$; and the space of all quadrics of $[n]$ is denoted Q_n. For example, in $[2]$, there is a $Q_2 = Q_2[5]$, every point of which is associated with a conic, which indeed is a $Q_2[0]$. In a quadric space any pair of distinct points generate a line, which corresponds to a pencil of quadrics of $[n]$ — a $Q_n[1]$.

A set of points of $[n]$ satisfying one homogeneous linear equation is called a prime (or hyperplane). It is also an $[n-1]$ contained in the given $[n]$. In a quadric space $Q_n[m]$, the subset of quadrics that pass through a given point P of $[n]$ satisfies

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one linear condition, and if not every quadric of $Q_n[m]$ passes through P then a prime of $Q_n[m]$ (i.e. a $Q_n[m-1]$) must do so. Extending this, in general k points determine k linear conditions on a quadric space for the quadrics to pass through the points. However, sometimes it is possible for the conditions to be dependent: later we shall see examples of this happening. In Hirschfeld's works [**11,13,14**], the important properties of quadrics in finite spaces can be found. The recent book by Hirschfeld and Thas [**15**], is also a good reference for these properties.

k-arcs are equivalent to certain types of M.D.S. codes, orthogonal arrays, and in special cases Laguerre planes and generalized quadrangles. A good reference to various similar types of problems in finite space is [**12**]. The many applications of k-arcs will not be mentioned after this point.

2. A Survey of Normal Rational Curves and Maximal Arcs

Definition 2.1. *A normal rational curve in $[n]$ is an irreducible curve of order n in $[n]$, which is not contained in any prime.*

It is well known that it may be represented (up to collineations of space) as the set of points

$$\{(1, x, x^2, \ldots, x^n) \mid x \in F \cup \{\infty\}\}.$$

Notice that the point with $x = \infty$ is just $(0, 0, \ldots, 1)$.

Definition 2.2. *A k-arc of $[n]$ $(k \geq n+1)$ is any set of k points of $[n]$ such that every $n+1$ points generate $[n]$. Equivalently, no $n+1$ points of an arc are linearly dependent (or contained in a prime).*

A basic problem of finite geometry is to construct large k-arcs of $PG(n,q)$, preferably with $k \geq q+1$. We always assume that $2 \leq n \leq q-2$ for the following reasons.

(1) If $n = 1$ then every subset of $PG(1,q)$ is an arc.
(2) It is an easy exercise to show that an $(n+3)$-arc exists in $PG(n,q)$ if and only if $1 \leq n \leq q-2$.
(3) There always exists an $(n+2)$-arc, which is equivalent by collineations to $\{e_0, e_1, \ldots, e_n, \sum_{i=0}^{n} e_i\}$, where e_i is the unit vector with all 0's except for a 1 in the i'th position.

Definition 2.3. *The orthogonal or Whitney dual of a k-arc $\mathcal{K}$ of $[n]$ is a k-arc $\mathcal{K}^d$ of $[k-n-2]$ defined as follows. $\mathcal{K}$ corresponds to a $(n+1) \times k$ matrix M over F of rank $n+1$, where the points of $\mathcal{K}$ are given by the columns of M. Let M^d be any $(k-n-1) \times k$ matrix over F of rank $k-n-1$, each row of which is orthogonal to each row of M. The columns of M^d correspond to the k-arc $\mathcal{K}^d$ which is the orthogonal dual of $\mathcal{K}$. It is easy to check that the 'dual' k-arc $\mathcal{K}^d$ is unique up to collineations and does not depend on the particular coordinate system that is used.*

The author likes to think of this type of duality (which is not the same as the one associated with polarities and correlations) as basically a matroid transformation. For the reader who is unfamiliar with matroid theory, there are many introductory books available: a good one is by Brylawski and Kelly; see [**3**]. Thus d can be thought of as a mapping from the matroid given by the set of points of $\mathcal{K}$ and the set of independent subsets of $\mathcal{K}$ to the dual matroid. A subset of points is a basis for the matroid (a maximal independent set) if and only if the complementary subset in the dual matroid (which has the same set of points) is also a basis for the dual. This idea is sometimes useful: for example, in matroid theory the concepts of deletion and contraction (i.e. deleting a point and projecting from a point) are dual to each-other. Thus the dual of the projection of $\mathcal{K}$ from a point P of $\mathcal{K}$ is isomorphic to $\mathcal{K}^d \setminus \{P\}$. This is true, both for general matroid isomorphism, and for projective space isomorphism (i.e. collineations). Note that a given matroid may be embeddable in the same projective space in various ways. For example, every k-arc has the same matroid structure, but there are usually many non-isomorphic k-arcs up to collineations. However, in some cases, the embedding is essentially unique; that is, all embeddings are collineation-isomorphic.

Let us prove a small result that will be needed in the next section. It is really a result that is valid for all matroids.

Lemma 2.4. *Let K be a set of k points of $[n]$ that generates the whole space and is not a k-arc. Then there exists a prime that contains n independent points of K and at least one point depending on them.*

Proof. K is a matroid with k points — actually a combinatorial geometry because there are no repeated points or *loops* (zero vectors). In it there are subsets called *circuits*, which are the minimal dependent sets. Since K is not an arc there will be a dependent set in K of size $n+1$ — this comes from some prime cutting the arc in at least $n+1$ points. Contained in this dependent set there will be a circuit C of size c satisfying $3 \leq c \leq n+1$. (Since K is a "geometry" each circuit has size at least 3.) Since K generates the whole space it will contain a basis B consisting of $n+1$ independent points. There is a fundamental property of all matroids, called the "basis exchange axiom", or something similar, that every independent set can be extended to a basis by adding in points from a given basis. Let $C = C' \cup \{d\}$ for some $d \in C$, where $C' = C \setminus \{d\}$. Then C' (of size $c-1$) is independent (since C is minimal), and can therefore be extended to a basis $B' \subseteq K$ by adding in points of B. Let b be any member of $B' \setminus C'$. (b exists since $c \leq n+1 \implies c-1 < n+1$.) Finally $B' \setminus \{b\} \cup \{d\}$ is the required subset of K, which lies in a prime. □

A normal rational curve is the classical example of a $(q+1)$-arc of $PG(n,q)$, although it is known that for the following cases there are other examples of k-arcs with $k \geq q+1$ in $PG(n,q)$.

Ovals. Many people call these *hyperovals*, but *oval* or $(q+2)$-*arc* was B. Segre's usage. In any case $n = 2$ and $k = q+2$, where $q = 2^h, (h \in \mathbb{N})$; and the orthogonal dual has $n = q-2$ and $k = q+2$.

(1) The first construction was that of an irreducible conic plus its *nucleus* (or *derived point* ... a term from algebraic geometry), and it certainly was known by the pioneers, e.g. R.C. Bose in the late 1930's.
(2) The first new constructions came from B. Segre, who in 1957 and 1962 discovered two basic types (with oval functions x^{2^i} where $(i,h) = 1$, and x^6

with h odd); see [**22,23**].

(3) In 1958, Lunelli and Sce had found a "sporadic" 18-arc in $PG(2,16)$ using computer-aided calculations; see [**16**].

(4) This was followed twenty years later by Glynn who discovered another two, (with oval functions $x^{3\sigma+4}$ and $x^{\sigma+\gamma}$, where $\sigma^2 \equiv \gamma^4 \equiv 2 \pmod{q-1}$, and h odd), see [**7**]. Note that both σ and γ are powers of 2, and so correspond to automorphisms of $GF(q)$.

(5) Then came another sequence by S.E. Payne [**18**], (still with h odd, with oval function $x^{\frac{1}{6}} + x^{\frac{1}{2}} + x^{\frac{5}{6}}$), that was related to elation generalized quadrangles and flocks of quadratic cones.

(6) In any case it is clear that the complete classification of all ovals will be hard, especially when other people, W. Cherowitzo, T. Penttila, etc., in the last few years have found other "sporadic" examples in small planes by using computer searches. See for example [**6,17,19**].

(7) At the 1993 Australasian Conference on Combinatorial Mathematics and Combinatorial Computing (in Adelaide) it was announced that two new infinite sequences of ovals had been constructed as a result of joint work in Perth by Cherowitzo, Penttila, Pinneri and Royle. These ovals are related to flocks of quadratic cones, and to elation generalised quadrangles of even order.

The $(q+2)$-arcs above induce, in the case $n = 2$, non-classical $(q+1)$-arcs in $PG(2,q)$ by the deletion of certain points; or by orthogonal duality, by projecting from a point of the arc, $(q+1)$-arcs of $PG(q-3,q)$. Thus, for $n = 2$ or $q-3$, and for $q = 2^h, k = q+1, (q \geq 8)$, there exist non-classical $(q+1)$-arcs. (The deletion of any point from a 6-arc of $PG(2,4)$ results in a conic.)

Segre Arcs. These are non-classical $(q+1)$-arcs of $PG(3,q)$, which have orthogonal duals in $PG(q-4,q)$. Their parameters are $n = 3$ or $q-4$, $q = 2^h$, $k = q+1$, $(q > 16, q \neq 64)$. They were constructed by B. Segre; see [**21**]. It was proved by L.R.A. Casse and the author that these examples are unique up to collineations; see [**4**]. The arcs are given by $\{(1, x, x^{2^i}, x^{2^i+1}) \mid x \in GF(q)\} \cup \{(0,0,0,1)\}$, where $(i,h) = 1$, and $i \neq 1$ or $h-1$.

The Non-classical 10-arc of PG(4,9). This, the only known non-classical arc of size at least $q+1$ in $PG(n,q)$, when q is odd, was discovered by the author in 1985; see [**8**]. It is self-dual (orthogonal duality), and its parameters are $n = 4, q = 9, k = 10$. The arc shall be investigated later in this paper.

Non-existence Results.

Although the intention of this paper is to stress constructive ideas — an existence proof is usually preferable to a non-existence proof (even though the latter could be called a "characterization theorem") — let us mention here very briefly that various authors, and in particular, J.A. Thas et al., have been able to prove the following, using algebraic methods, that are an extension to Segre's "Lemma of Tangents", which was used for odd q to classify the $(q+1)$-arcs in $PG(2,q)$ (see [**20**]), and in $PG(3,q)$ (see [**21**]). (They are all normal rational curves.)

Remark 2.5.

(1) *There are no $(q+2)$-arcs of $PG(n,q)$ if $n \geq 3$ is fixed and q is larger than some known function of n. Also, if q is large enough every $(q+1)$-arc is a normal rational curve. The bounds are better for q even than for q odd. See* [**27**], *which also contains other references.*

(2) *There are no new $(q+1)$-arcs to be discovered in $PG(n,q)$, when $n \leq 3$, if q is odd (see* [**20,21**]*); or when $n \leq 4$, if q is even (see* [**4,5**]*).*

(3) *The smallest unsolved cases appear to be in $PG(4,11)$ and $PG(5,16)$. (The 10-arcs of $PG(4,9)$ were classified in* [**8**]*.)*

Note that from orthogonal duality the case of $(q+1)$-arcs in $PG(n,q)$ is equivalent to the case of $(q+1)$-arcs in $PG(q-n-1,q)$. At present not much is known about $(q+1)$-arcs in spaces of dimension about $q/2$, where $q \geq 11$.

3. Intersections of quadrics

Let the points of $[n]$ be

$$(x_0, x_1, \ldots, x_n),$$

using homogeneous coordinates over F. A *quadric* of $[n]$ is the set of points satisfying an equation of type

$$\mathcal{Q}: \sum_{0 \leq i \leq j \leq n} a_{ij} x_i x_j = 0, \text{ for fixed } a_{ij} \in F, \text{ (not all zero).}$$

Thus the number of parameters a_{ij} of a quadric in $[n]$ is $\binom{n+2}{2}$, so that there is a $Q_n = Q_n[\binom{n+2}{2} - 1]$ of quadrics in $[n]$.

Theorem 3.1. *Consider a set S of quadrics of $[n]$ generated by a collection of $\binom{n}{2}$ independent quadrics. (Thus S is a $Q_n[\binom{n}{2} - 1]$.) Let $\mathcal{K} := \cap S$ be the set of points of $[n]$ common to all quadrics of S. Suppose that $\mathcal{K}$ generates $[n]$ and that S does not contain any quadric that is the union of two primes (hyperplanes) of $[n]$. Then $\mathcal{K}$ is an arc of $[n]$.*

Proof. Suppose $\mathcal{K}$ is not an arc but satisfies the conditions of the theorem. We can use Lemma 2.4. There exists a prime h containing n independent points of $\mathcal{K}$ and one point of $\mathcal{K}$ dependent upon them. We can assume that these $n+1$ points are the unit vectors e_i, for $0 \leq i \leq n-1$, and $(d_0, \ldots, d_{n-1}, 0)$, where $d_i d_j \neq 0$ for some $i \neq j$. Then a general quadric $\sum_{0 \leq i \leq j \leq n} \lambda_{ij} x_i x_j = 0$ passes through these points if and only if $\lambda_{ii} = 0, \forall 0 \leq i \leq n-1$, and $\sum_{0 \leq i \leq j \leq n-1} \lambda_{ij} d_i d_j = 0$. Thus we obtain $n+1$ independent linear conditions in the space $Q_{n-1}[\binom{n+1}{2} - 1]$ of all quadrics in h. Hence the dimension of the set of all quadrics of $h = [n-1]$ containing these $n+1$ points is

$$\binom{n+1}{2} - 1 - (n+1) = (n^2 + n - 2 - 2n - 2)/2 = \binom{n}{2} - 2.$$

However, the condition that S does not contain any quadric that is the union of two primes (hyperplanes) implies that no two quadrics Q_1 and Q_2 of S are the same on h. Otherwise their difference, $Q_1 - \lambda Q_2$ for some $\lambda \in F$, would be a quadric of type $h.h'$, for some prime $h' \in [n]$. Hence $S \cap h$ is a space of quadrics of the same dimension $\binom{n}{2} - 1$ as S, and this gives a contradiction. $\square$

It remains to show that there are spaces of quadrics that satisfy the conditions of the above theorem. Consider now $Q_n = Q_n[\binom{n+2}{2} - 1]$. The set of all *cones* (in other words singular quadrics) of $[n]$ is a primal (or hypersurface) Δ in Q_n of order $n+1$. (If F is not of even characteristic then $\sum_{0 \le i \le j \le n} a_{ij}x_i x_j = 0$ can be rewritten $\sum_{0 \le i \le n,\ 0 \le j \le n} b_{ij}x_i x_j = 0$, where $b_{ii} = a_{ii}, \forall\, i$, and $b_{ij} = b_{ji} = \frac{1}{2}a_{ij}, \forall\, i < j$. The condition that the quadric is singular is $\det(b_{ij}) = 0$, which has degree $n+1$ since (b_{ij}) is an $(n+1) \times (n+1)$ matrix. Similarly, when the characteristic of F is 2, there is another condition of degree $n+1$ in the coefficients; see [13].)

In Δ there are various subvarieties corresponding to different spaces of singular quadrics. In particular, the collection of unordered prime-pairs is an algebraic variety V_{2n}^m of dimension $2n$ and some order m. (The dimension is $2n$ because the degree of freedom of each hyperplane is n.) The order m of a variety V_α^m is the finite number of points in which a general $[n-\alpha]$ intersects it (over an algebraically closed extension of F). Now the dimension of Q_n is

$$\binom{n+2}{2} - 1 = \binom{n}{2} + 2n,$$

and so m is the number of points that a general $Q_n[\binom{n}{2}]$ meets V_{2n}^m. By the specialization principle [26], we can assume that the $Q_n[\binom{n}{2}]$ is the space of quadrics through $2n$ general points. In this space there are $\binom{2n}{n}/2$ unordered prime-pairs. Hence

$$m = \binom{2n}{n}/2.$$

As an example, consider the case $n = 2$. Then the conics of the plane form a $Q_2 = Q_2[5]$. In this space the cones correspond to the cubic primal, $\Delta = V_4^3$. However, the variety of line-pairs, a $V_{2n}^{\binom{2n}{n}/2} = V_4^3 = \Delta$. This happens because every singular conic in the plane splits into the product of two lines. In the case $n = 3$, however, the plane-pairs of [3] produce a proper subvariety of Δ. In fact we get a $V_6^{10} \subset V_8^4 = \Delta$ in $Q_3 = Q_3[9]$.

Getting back to our previous problem, we must find a $Q_n[\binom{n}{2} - 1]$ that is skew to the $V_{2n}^{\binom{2n}{n}/2}$. From the dimensions of the varieties we see that this is indeed the general case: varieties are generally skew if the sum of their dimensions is less than the dimension of the ambient space. In particular, the space of quadrics through $2n+1$ general points should not intersect V_{2n}^m, (if the field F is large enough). Choosing $n+1$ of these points to be linearly independent then gives a k-arc, by Theorem 3.1. Of course, the word "general" must mean in this case that the $2n+1$ points form an arc, and so this argument is circular.

However, a method that could be used to extend a $2n$-arc A to a larger arc A' is then obvious. We consider the space of quadrics $Q_n[\binom{n}{2}]$ through A. This space should contain m quadrics which are prime-pairs. If the size of the field is large enough, there will be primes of $Q_n[\binom{n}{2}]$, or equivalently $Q_n[\binom{n}{2} - 1]$'s of Q_n, that satisfy the conditions of Theorem 3.1, and which give spaces of quadrics intersecting in arcs containing A.

Theorem 3.2.

(1) *Every $(2n+1)$-arc of $[n]$ is in the intersection of a unique $Q_n[\binom{n}{2} - 1]$ of quadrics that doesn't contain a prime-pair.*

(2) *Every k-arc of $[n]$ with $k \leq 2n+1$ is in the intersection of a unique $Q_n[\binom{n+2}{2} - 1 - k]$ of quadrics that contains prime-pairs if $k \neq 2n+1$.*

Proof. It is clear that a $(2n+1)$-arc is not contained in a prime-pair because each prime contains at most n points of the arc. Hence there are fewer quadrics passing through the $(2n+1)$-arc than through any of its sub-$2n$-arcs. Each point determines a linear condition on the set of quadrics of $[n]$, and so the $2n+1$ conditions corresponding to the points of the $(2n+1)$-arc are independent. From the previous discussion part (1) is proved. The argument is similar for $k \leq 2n$. In this case there is always a prime-pair passing through a sub-$(k-1)$-arc that doesn't pass through the k-arc. Hence the k conditions given by the points of such a k-arc are independent. The rest follows from the fact that the dimension of the space Q_n is $[\binom{n+2}{2} - 1]$. □

Theorem 3.3. *A normal rational curve of $[n]$ is the complete intersection of a space S of quadrics $Q_n[\binom{n}{2} - 1]$ not containing a prime-pair. If F is infinite, or if $F = GF(q)$, where $q \geq 2n$, every sub-$(2n+1)$-arc of the curve determines the same set of quadrics. If $F = GF(q)$ and $q < 2n$, then the space of all quadrics, including prime-pairs, that passes through the arc, is a $Q_n[\binom{n+2}{2} - 2 - q]$.*

Proof. For $n = 2$ a normal rational curve is a non-degenerate quadric (conic), which is a $Q_2[0]$. This generalises to any dimension $n \geq 2$ as follows. Let the curve in $[n]$ be

$$\{(1, x, x^2, \ldots, x^n) \mid x \in F \cup \{\infty\}\}.$$

The space generated by the quadrics

$$x_i x_j = x_{i+1} x_{j-1}, \text{ for } 0 \leq i \leq j-2,\ 2 \leq j \leq n$$

is of the type needed by Theorem 3.1. Also, Theorem 3.2(1) implies that every sub-$(2n+1)$-arc lies in the intersection of a unique $Q_n[\binom{n}{2} - 1]$. The condition $|F| \geq 2n$ is necessary because the number of points on a normal rational curve is $|F| + 1 = q + 1$. Finally, if $q < 2n$, then Theorem 3.2(2) gives the dimension of the space of all quadrics passing through the normal rational curve. □

Theorems 3.1 and 3.3 give a way to prove that every normal rational curve is an arc. Note that the above shows that any $2n+2$ points of a normal rational curve only determine $2n+1$ conditions on the quadric space. Thus the $2n+1$

of Theorem 3.2 is the largest possible. In fact, if $q \geq 2n$, the points of a normal rational curve correspond to a $(q+1)$-arc of the "dual condition space" of dimension $2n$. The dual condition space could be constructed as follows. The quadrics of Q_n that pass through a fixed point of $[n]$ correspond to a certain prime of Q_n. The "dual condition space" is the set of all primes of Q_n — these can be considered to be points of a dual projective space (with the usual type of projective geometry duality.) Hence the $q+1$ points of the normal rational curve correspond to a set of $q+1$ points of a $[2n]$ contained in the dual condition space. The reader can check (it is not hard) that it is another normal rational curve.

Theorem 3.4. *Let $\mathcal{K}$ be the intersection of a general quadric space $S = Q_n[e]$ of dimension e in $[n]$. Let P be a point of $[n]$ and let $\mathcal{K}_P$ be the projection of $\mathcal{K}$ from P to an $[n-1]$ not through P. Then $\mathcal{K}_P$ is contained in the intersection of an $S_P = Q_{n-1}[e']$, where $e' \geq e - n$ if $P \in \mathcal{K}$, or $e' \geq e - n - 1$ if $P \notin \mathcal{K}$.*

Proof. The space of cones with vertex P that contain $\mathcal{K}$ is in 1-1 correspondence with the space of quadrics of the $Q_{n-1}[e']$ that contain $\mathcal{K}_P$. If we coordinatise so that $P = (0, 0, \ldots, 0, 1)$, then the cones with vertex P are those quadrics not having terms in $x_i x_n$, for $i = 0, \ldots, n$. Therefore we have $n+1$ linear conditions that a quadric of Q_n is such a cone. If these are independent (which may occur if $P \notin \mathcal{K}$) then this implies that $e' = e - n - 1$. However, if they are dependent we can say that the dimension e' is larger than this. If however, $P \in \mathcal{K}$, we already know that there is no term x_n^2 because every quadric in S passes through P. Hence there are n linear conditions that a quadric of S is a cone with vertex P. If these are independent we see that S_P is a quadric space of dimension $e - n$. Otherwise it has higher dimension. □

Example 3.5. *Projections of normal rational curves*

If we project a normal rational curve $\mathcal{K}$ of $[n]$ from one of its points, we obtain an arc of $[n-1]$ that is contained in another (in general unique) normal rational curve. (W.l.o.g. the point can be taken to be e_n, and the curve to be $(1, x, \ldots, x^n)$, $x \in F \cup \{\infty\}$. The projection is then $(1, x, \ldots, x^{n-1})$, where $x \in F$.) By Theorem 3.3, if $q \geq 2n$ or $|F|$ is infinite there is a $Q_n[\binom{n}{2} - 1]$ of quadrics containing $\mathcal{K}$ and also a $Q_{n-1}[\binom{n-1}{2} - 1]$ of quadrics containing $\mathcal{K}_P, \forall P \in \mathcal{K}$. Since

$$\binom{n}{2} - 1 - (\binom{n-1}{2} - 1) = n - 1,$$

we see that the n conditions for a quadric of S to be a cone with vertex P are dependent and that they actually are only of rank $n-1$.

4. Properties of the Non-classical 10-arc of PG(4,9)

Let $F = GF(9)$ and let $n \in F$ where $n^4 = -1$, (i.e. n is a non-square of F). The 10-arc, (see [**8**]), is given by the set of points

$$\mathcal{K} := \{P_x \mid x \in F\} \cup \{(0,0,0,0,1)\}, \text{ where } P_x := (1, x, x^2 + nx^6, x^3, x^4), \ \forall x \in F.$$

There is also another special point, $N := (0,0,1,0,0)$ of $PG(4,9)$, associated with $\mathcal{K}$. In this section we investigate the quadrics that contain $\mathcal{K}$.

Some of the properties, as proved in [**8**], were:

Theorem 4.1.

(1) *The full group of collineations fixing the 10-arc is isomorphic to $PGL(2,9)$, and acts sharply 3-transitively on the 10 points. (Thus the group has order 10.9.8 = 720.) This group fixes the special point N.*

(2) *The projection of the arc from N is an elliptic quadric embedded in a Baer subgeometry $PG(3,3)$ of $PG(3,9)$.*

Now we can show more.

Theorem 4.2. *The space of quadrics passing through $\mathcal{K}$ is generated by*

$$\begin{aligned} \mathcal{Q}_0 &: nx_1^2 + x_3^2 - x_2x_4 = 0, \\ \mathcal{Q}_1 &: x_2x_3 - nx_0x_1 - x_1x_4 = 0, \\ \mathcal{Q}_2 &: x_1x_3 - x_0x_4 = 0, \\ \mathcal{Q}_3 &: x_1x_2 - x_0x_3 - nx_3x_4 = 0, \\ \mathcal{Q}_4 &: x_1^2 + nx_3^2 - x_0x_2 = 0. \end{aligned}$$

Thus it is a $Q_4[4]$. Each of these quadrics also passes through the special point N.

Proof. Let $(x_0, \ldots, x_4)$ be in the intersection of the $Q_4[4]$ above. If $x_0 = 0$, we have from $\mathcal{Q}_2$ that

$$x_1x_3 = 0 \Rightarrow x_1 = 0 \text{ or } x_3 = 0.$$

From $\mathcal{Q}_4$ we have that both x_1 and x_3 are zero, and then $\mathcal{Q}_0$ implies that $x_2x_4 = 0$. This gives us the two points $(0,0,0,0,1)$ and $(0,0,1,0,0)$. Now suppose $x_0 = 1$. Then $\mathcal{Q}_2$ implies that $x_4 = x_1x_3$, while $\mathcal{Q}_4$ implies that $x_2 = x_1^2 + nx_3^2$. Substituting these into the other three equations implies that

$$\begin{aligned} &(1) \qquad & nx_1^2 + x_3^2 - x_1^3x_3 - nx_1x_3^3 &= 0, \\ &(2) \qquad & x_1^2x_3 + nx_3^3 - nx_1 - x_1^2x_3 &= 0, \\ &(3) \qquad & x_1^3 + nx_1x_3^2 - x_3 - nx_1x_3^2 &= 0. \end{aligned}$$

Since $(1) + x_1(2) \Rightarrow x_3^2 - x_1^3x_3 = 0$; and $(1) + x_3(3) \Rightarrow nx_1^2 - nx_1x_3^3 = 0$, then either $x_1 = x_3 = 0$, or else $x_1 = x_3^3, x_3 = x_1^3 \Rightarrow x_1 = x_1^9$. Hence $x_1 \in GF(9)$. This soon gives the other nine points of $\mathcal{K}$.

Consider the equation of a general quadric $\mathcal{Q}$ with coefficients (a_{ij}) $(0 \le i \le j \le 4)$ passing through $\mathcal{K}$. The point $(0,0,0,0,1)$ implies that $a_{44} = 0$, while the other points of $\mathcal{K}$ imply the equations $\mathcal{Q}(P_x) = 0, \forall x \in F$. Hence

$$\mathcal{Q}(P_x) = a_{00} + a_{01}x + \cdots + a_{22}(x^2 + nx^6)^2 + \cdots + a_{44}x^8 \equiv 0 \pmod{x^9 - x}.$$

The coefficient of x^8 in $\mathcal{Q}(P_x)$ is $2na_{22} + a_{44}$. Thus a_{22} is zero and $\mathcal{Q}$ also passes through N. Notice that the equations arising from the condition that $\mathcal{Q}$ passes through $\mathcal{K}$ are independent — there are 10 of them. Hence the dimension of the space of quadrics through $\mathcal{K}$ is

$$\binom{4+2}{2} - 1 - 10 = 4.$$

Thus the above quadrics do generate all the quadrics passing through $\mathcal{K}$. $\square$

Any quadric not passing through N contains a term x_2^2, and in particular

$$\mathcal{Q}_5 : x_2^2 = (n^2+1)x_0x_4 + 2nx_4^2$$

does not contain N. However, it can be checked that this latter quadric contains the 9 points of $\mathcal{K}\setminus\{(0,0,0,0,1)\}$. Let S be the $Q_4[5]$ generated by all six of the quadrics $\mathcal{Q}_0, \ldots, \mathcal{Q}_5$ above. Then $\cap S$ is the 9-arc that is $\mathcal{K}\setminus\{(0,0,0,0,1)\}$ (which spans [4]). This intersection cannot contain more points because, by Theorem 3.1, if it did $\cap S$ would be a 10-arc, which would be either a normal rational curve, or of non-classical type. However, the non-classical 9-arc of $PG(4,9)$ can only be extended to a non-classical 10-arc (its orthogonal dual is the complete 9-arc of $PG(3,9)$), while the non-classical 10-arc is not in the intersection of a $Q_4[5]$.

Now S does not contain any prime-pairs — note that $9 = 2.4+1$, and use Theorem 3.1; or one can prove it directly. In any case Theorem 3.1 applies to S, verifying that $\cap S$ is a 9-arc. The automorphism group of $\mathcal{K}$ is sharply 3-transitive by Theorem 4.1 and hence transitive on the 10 points. Since any subset of 5 points of the above 9-arc is independent, one can use the group to shift any subset T of 5 points of $\mathcal{K}$ to a subset of the 9-arc. (Just map the complement of T to a subset containing $(0,0,0,0,1)$.) Thus the group can be used to prove that $\mathcal{K}$ is a 10-arc. In any case, it is interesting that the 9-arc of $PG(4,9)$ that is the intersection of the $Q_4[5]$ of quadrics above, is the orthogonal dual of the complete 9-arc of $PG(3,9)$. Thus a construction of this complete 9-arc from quadrics is obtained.

Next we consider the implications of Theorem 3.4 with respect to the quadric space $S = Q_4[4]$ through $\mathcal{K}\cup\{N\}$. Let P be any point of $\mathcal{K}\cup\{N\}$. Then there is a [0] of cones (a unique cone) of S_P. Thus the four conditions for a quadric of S to be a cone with vertex P are independent. In the case of $P = N$, the cone of S_P is

$$x_0x_4 = x_1x_3,$$

which is hyperbolic, while if P is any point of the 10-arc, say $(0,0,0,0,1)$, the cone of S_P is

$$x_0x_2 = x_1^2 + nx_3^2,$$

which is elliptic.

The cones contained in the $Q_4[4]$ actually correspond to the quintic primal which is the intersection of the quintic primal Δ of §3 with $Q_4[4]$. This contains a configuration isomorphic to $\mathcal{K}\cup\{N\}$. In fact, we have defined $\mathcal{Q}_0,\ldots,\mathcal{Q}_4$ so that the collineation from the point [4] to $Q_4[4]$ mapping $(\lambda_0,\ldots,\lambda_4)\mapsto\sum_{i=0}^{4}\lambda_i\mathcal{Q}_i$ takes a point of $\mathcal{K}\cup\{N\}$ to the cone with *that point as vertex*; see Theorem 4.6. Thus $\mathcal{K}$

could be constructed from the elliptic cones of a $Q_4[4]$ and/or the above quadratic mapping.

The non-classical 10-arc $\mathcal{K}$ of $PG(4,9)$ has many other interesting properties that can be deduced principally from the [4] of quadrics passing through $\mathcal{K}$ and the special point N. Before we start this consider the following.

Definition 4.3. *A derived point of a primal $f(x) = 0$ of order d contained in $[n]$ over a field F is any point such that all the $n+1$ partial derivatives of f are zero at that point. A singularity of f is a derived point that lies on the primal.*

Now Euler's formula states that

$$\sum_{i=0}^{n} x_i . \frac{\partial f}{\partial x_i} = d.f(x).$$

Hence there can only be derived points that are not singularities if the characteristic of the field F divides d. The first interesting case is that of conics in projective planes of characteristic 2, where the derived point is called the *nucleus*.

Here is a curious property of the normal rational curve of $PG(4,9)$.

Theorem 4.4. *The normal rational curve $\{(1,x,x^2,x^3,x^4) \mid x \in GF(9) \cup \{\infty\}\}$ together with the point $N = (0,0,1,0,0)$, is the set of derived points of any cubic primal of the form*

$$\mathcal{D} : x_0x_3^2 - x_0x_2x_4 + x_1x_2x_3 + x_1^2x_4 + \sum_{i=0}^{4} \alpha_i x_i^3 = 0,$$

where $\alpha_i \in GF(9)$.

Proof. By Theorem 3.3 the quadric space, whose intersection is the normal rational curve, is generated by the quadrics

$$x_ix_j = x_{i+1}x_{j-1}, \text{ for } 0 \le i \le j-2, \ 2 \le j \le 4,$$

which are therefore

$$x_2x_4 = x_3^2; \ x_1x_4 = x_2x_3; \ x_0x_4 = x_1x_3; \ x_1x_3 = x_2^2; \ x_0x_3 = x_1x_2; \ x_0x_2 = x_1^2.$$

Only one of these quadrics does not pass through N: it is $x_1x_3 = x_2^2$. The others have as their complete intersection the normal rational curve and the point N. Also, the partial derivatives of $\mathcal{D}$ are the five quadrics passing through N above. Hence the cubic has precisely 11 derived points which are those on the normal rational curve, and also the point N. We leave some of the details because next we show that a similar thing happens for the non-classical arc. □

Theorem 4.5. *$\mathcal{K} \cup \{N\}$ is the set of derived points of a cubic primal of $PG(4,9)$*

$$C: nx_0x_1^2 + x_0x_3^2 - x_0x_2x_4 + x_1x_2x_3 + x_1^2x_4 + nx_3^2x_4 + \sum_{i=0}^{4} \alpha_i x_i^3 = 0,$$

where $\alpha_i \in GF(9)$.

Proof. It is only necessary to note that

$$\frac{\partial C}{\partial x_i} = \mathcal{Q}_i, \quad \forall\, 0 \le i \le 4.$$

From Theorem 4.2, $\mathcal{K} \cup \{N\}$ is the intersection of these quadrics and so $\mathcal{K} \cup \{N\}$ is the set of derived points of C. Note that some of these points can lie on the cubic primal, depending on how the α_i's are chosen. □

Some of the properties of the space of quadrics containing the non-classical 10-arc $\mathcal{K}$ of $PG(4,9)$ are really related to general properties of cubic primals in any projective space $[n]$. Given a cubic primal f of $[n]$ there is always a space $Q_n[n]$ of first-polar quadrics associated with f. The first-polar of a point $(y_0, \ldots, y_n)$ is the quadric $\sum_{i=0}^{n} y_i \frac{\partial f}{\partial x_i} = 0$. The Hessian matrix of f is the $(n+1) \times (n+1)$ matrix of all second-order partial derivatives

$$H := \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right).$$

When f is cubic, H is linear in the variables x_i, and this provides a connection, via an identity, to the polar quadrics of f.

Theorem 4.6. *Let f be any cubic primal of $[n]$, where the base field does not have characteristic two. Then the following hold.*

(1) *There is an identity valid for all variables x_i, y_j.*

$$(y_0, \ldots, y_n) H (y_0, \ldots, y_n)^t = 2 \sum_{i=0}^{n} x_i \frac{\partial f}{\partial y_i}.$$

Thus the symmetric matrix of the polar quadric (in $Q_n[n]$) of the point $(x_0, \ldots, x_n)$ is given by evaluating the Hessian matrix at this point.

(2) *Another identity in x_i is:*

$$(x_0, \ldots, x_n) H = 2\left(\frac{\partial f}{\partial x_0}, \ldots, \frac{\partial f}{\partial x_n}\right).$$

Thus a point is a vertex of its own polar quadric if and only if it is a derived point of f.

(3) *The quadratic point to hyperplane transformation τ of $PG(4,9)$ associated with the 10-arc $\mathcal{K}$, defined (in dual coordinates) by*

$$\tau: P \mapsto [\mathcal{Q}_0(P), \mathcal{Q}_1(P), \mathcal{Q}_2(P), \mathcal{Q}_3(P), \mathcal{Q}_4(P)] \text{ satisfies}$$

$$\tau(P) = P\sum_{i=0}^{4} p_i A_i, \text{ where}$$

$P = (p_0, \ldots, p_4)$ and A_i is the symmetric 4×4 matrix corresponding to $\mathcal{Q}_i$.

(4) *With respect to the 10-arc $\mathcal{K}$ of $PG(4,9)$ we have: $(\lambda_0, \ldots, \lambda_4) \in \sum_{i=0}^{4} \lambda_i \mathcal{Q}_i$, $\forall$ points of $PG(4,9)$. Thus every point is contained in its polar quadric. Further, a point is the vertex of the cone which is its polar quadric if and only if it is a derived point; that is, it lies on $\mathcal{K} \cup \{N\}$.*

Proof. (1) The cubic primal f contains three types of terms: $x_ix_jx_k$, (i, j, k distinct); $x_ix_j^2$, (i, j distinct); and x_i^3. If the identity is valid for these terms then it is valid, by summation, for all cubics. Thus in the first case we obtain, on both sides the form $2(y_iy_jx_k + y_ix_jy_k + x_iy_jy_k)$; and in the second the form $2(2y_iy_jx_j + x_iy_j^2)$; and in the third $6x_iy_i^2$.
(2) This follows immediately from Euler's identity, since a column of H is the vector of first partial derivatives of a primal of degree 2. The second assertion follows from the fact that the vertices of a quadric cone correspond to the non-zero vectors of the null space of the symmetric matrix (in this case H) corresponding to the quadric; see [**13**].
(3) This follows directly from (1) and (2), when we realize that $\mathcal{Q}_i = \frac{\partial \mathcal{C}}{\partial x_i}$ is the polar quadric of the i'th unit vector e_i, and so its matrix A_i (up to multiplication by 1/2) is obtained by evaluating H at e_i.
(4) This comes from Euler's identity in the case of one of the cubic primals $\mathcal{C}$ associated with $\mathcal{K}$, since the characteristic of $GF(9)$ is three, which divides the order, 3, of $\mathcal{C}$. □

At first the author thought that there was some connection between the non-classical 10-arc of $PG(4,9)$, and the theory of quintic curves and Segre 10-nodal cubic primals of four dimensional space ... perhaps because the number 10 was involved in both cases; see [**26, Chapter VIII**, §5], where many properties of the elliptic quintic and its related quadro-cubic Cremona transformation are listed. See also [**25**], where "the most beautiful of all quadric transformations" is described in detail, and also [**1**]. Let us discuss these things for a while. There are two types of irreducible quintic curves of [4].

(1) *Elliptic*: take three quadrics of [4] (generating a $Q_4[2]$) that intersect in a normal rational cubic curve of a [3] and residually in an elliptic quintic. It is explained in [**25,26**] how the $Q_4[4]$ through this curve determines a quadratic Cremona transformation with cubic inverse. Associated also with this curve are Segre cubic primals of [4], which have 10 nodes. However, there are 15 planes contained in such a primal, each containing 4 nodes. Hence the set of nodes is not a 10-arc, and so our cubic primal $\mathcal{C}$ containing $\mathcal{K} \cup \{N\}$ does not appear to be a Segre cubic primal.
(2) *Rational*: although a rational curve is a special case of an elliptic we can find another way to construct a $Q_4[4]$ through any rational quintic curve of [4]. Consider the following. Every rational curve A of order $n+1$ in $[n]$ is obtained by projecting a normal rational curve R of $[n+1]$ from a point

$P \notin R$. There is a $Q_{n+1}[\binom{n+1}{2}-1]$ through R and so the dimension of the space of cones with vertex P containing R is at least

$$\binom{n+1}{2} - 1 - (n+1) = \binom{n}{2} - 2.$$

(See Theorem 3.4.) Hence the rational curve A is contained in a $Q_n[\binom{n}{2}-2]$. Now put $n = 4$ to obtain the result for rational quintic curves of $[4]$.

Thus both the non-classical 10-arc, and the quintic curves above have a $Q_4[4]$ through them. Despite this we have the following.

Theorem 4.7. *The non-classical 10-arc $\mathcal{K}$ of $PG(4,9)$ together with its special point N are not contained in any irreducible quintic curve.*

Proof. $\mathcal{K} \cup \{N\}$ is the precise intersection of the $Q_4[4]$ of quadrics generated by $\mathcal{Q}_0, \ldots, \mathcal{Q}_4$; see Theorem 4.2. Since the $Q_4[4]$ through the eleven points is unique, and since any quintic curve has an infinite number of points over the algebraic closure of $GF(9)$, we see that the set of intersection of the quadrics is not a quintic curve. It is also not contained in any rational or elliptic (or irreducible) quintic curve of $[4]$ — the $Q_4[4]$ containing the quintic curve and that containing $\mathcal{K}$ could not be the same, and so there would be at least a $Q_4[5]$ containing $\mathcal{K}$ — a contradiction. □

Theorem 4.8. *The chords of $\mathcal{K} \cup \{N\}$ are contained in a quintic primal with the property that each point of the chord is the vertex of a single cone or pencil of cones of the $Q_4[4]$ of quadrics containing $\mathcal{K} \cup \{N\}$. There are two points (possibly imaginary), on each chord of $\mathcal{K} \cup \{N\}$ which are vertices of a pencil of quadrics.*

Proof. If A_i is the symmetric matrix of $\mathcal{Q}_i$, then

$$det\left(\sum_{i=0}^{4} p_i A_i\right) = 0$$

is the equation of the set of cones of the space of quadrics. This is a quintic primal. From Theorem 4.6(4) it follows that $P = (p_0, \ldots, p_4)$ is the vertex of the cone above if and only if $P \in \mathcal{K} \cup \{N\}$. Consider two points of $\mathcal{K}$, which we may take to be $(1,0,0,0,0)$ and $(0,0,0,0,1)$ by the 3-transitivity of the group of $\mathcal{K}$. A general point on the line joining these points is $(1,0,0,0,k)$, where $k \neq 0$. The quadric

$$\sum_{i=0}^{4} p_i A_i = \begin{pmatrix} 0 & np_1 & p_4 & p_3 & p_2 \\ np_1 & np_0 + p_4 & -p_3 & -p_2 & p_1 \\ p_4 & -p_3 & 0 & -p_1 & p_0 \\ p_3 & -p_2 & -p_1 & p_0 + np_4 & np_3 \\ p_2 & p_1 & p_0 & np_3 & 0 \end{pmatrix},$$

is a cone with vertex $(1,0,0,0,k)$ if and only if $(1,0,0,0,k)\sum_{i=0}^{4} p_i A_i = 0$. Solving this we find that

$$p_2 = (n+k)p_1 = p_4 + kp_0 = (1+nk)p_3 = 0.$$

These equations show that there is a unique cone through all but two points on the chord joining any two points of $\mathcal{K}$. The two points with a pencil of cones are

$$(1,0,0,0,-n) \text{ and } (-n,0,0,0,1).$$

Similarly, each point on the line joining $(1,0,0,0,0)$ and $(0,0,1,0,0)$ is the vertex of a unique cone, except for two points

$$(1,0,k,0,0), \text{ where } k^2 = n,$$

which in this case are in $PG(4,81)$, as n is a non-square of $GF(9)$. $\square$

Note that this property is very similar to that of a quintic curve of [4]. See [**26**].

5. Conclusions

We have given some of the basic theory connecting quadrics and k-arcs, so here is the place to include some conjectures and ideas about further progress in the subject.

Conjecture. *A $(q+1)$-arc of $PG(n,q)$ that is contained in a $Q_n[\binom{n}{2}-1]$ of quadrics is a normal rational curve. One may need to assume that $q \geq 2n$.*

Note that if this could be proven then every $(q+1)$-arc of $PG(q/2,q)$, where $q = 2^h$, would be a normal rational curve. This is because of Theorem 3.2(1). By orthogonal duality, every $(q+1)$-arc of $PG((q-2)/2,q)$ would also be a normal rational curve. The first counter-example might be in $PG(8,16)$.

Conjecture. *There exist more examples of q-arcs of $PG(n,q)$ that are the complete intersection of a $Q_n[\binom{n}{2}-1]$ of quadrics.*

The non-classical 9-arc of $PG(4,9)$ contained in the 10-arc $\mathcal{K}$ is an example.

Conjecture. *The maximum dimension of a space of quadrics of $[n]$ that does not contain a prime-pair is $\binom{n}{2}-1$. Note that by dimension theory this is true for geometries over algebraically closed fields.*

Conjecture. *For most $q \geq 2n$ there exist examples of $(2n+1)$-arcs of $[n]$ not contained in a normal rational curve: by Theorem 3.2, these are the intersection of a $Q_n[\binom{n}{2}-1]$ of quadrics.*

For $n=2$, every $(2n+1)$-arc of $[n]$ is in a unique quadric (conic). For $n=3$, a 7-arc of [3] is orthogonally dual to a 7-arc of [2]. There are plenty of such arcs not contained in a conic. In fact, they have all been counted [**9, Theorem 4.2**]. Hence the number of 7-arcs of $PG(3,q)$ is given by a similar formula involving one variable, noting the result by Thas [**29**], which essentially says if K_n is the number of k-arcs in $[n]$, and if H_n is the number of homographies (linear collineations) in $[n]$, then

$$H_{k-n-2}K_n = H_nK_{k-n-2}.$$

This is best thought of as a relation between the orthogonal duals.

Main Problem. *Generalise the non-classical* 10-*arc of* $PG(4,9)$.

One could look at curves (elliptic or rational) of order $n+1$ in $[n]$. Another idea is to consider the space of quadrics of a normal rational curve of $[n+1]$, and then construct good spaces of quadrics of $[n]$ by projection or by restriction to a hyperplane. The first cases to consider in $PG(n,q)$ would be q odd, $n \geq 4$, $q \geq 11$; or q a power of 2, $n \geq 5$, $q \geq 16$. One could also look at sets of derived points of certain primals to see if arcs are hidden in them.

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