AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **90(2)** (2024), Pages 187–198

On the number of generators of groups acting arc-transitively on graphs

Marco Barbieri

Dipartimento di Matematica "Felice Casorati"
University of Pavia
Via Ferrata 5, 27100 Pavia, Italy
marco.barbieri07@universitadipavia.it

Pablo Spiga

Dipartimento di Matematica e Applicazioni
University of Milano-Bicocca
Via Cozzi 55, 20125 Milano, Italy
pablo.spiga@unimib.it

Abstract

Given a connected finite graph Γ and a group G acting transitively on the vertices of Γ, we prove that the number of vertices of Γ and the order of G are bounded above by a function depending only on the valency of Γ and on the exponent of G. We also prove that the number of generators of a group G acting transitively on the arcs of a locally finite graph Γ cannot be bounded by a function of the valency alone.

1 Introduction

A *vertex-transitive graph* is a pair (Γ, G) where Γ is a locally finite connected graph and G is a subgroup of $\mathrm{Aut}(\Gamma)$ whose action on the vertex-set of Γ is transitive. In this note, we assume that, for every vertex α of Γ, the order of the vertex-stabilizer G_α is finite. The *local group of* (Γ, G) is the permutation group that a vertex-stabilizer G_α induces on the neighbourhood $\Gamma(\alpha)$ of the fixed vertex α. In particular, the degree of the local group of (Γ, G) coincides with the *valency of* Γ. We say that a vertex-transitive graph (Γ, G) is *arc-transitive* if the local group of the pair (Γ, G) is transitive. As the name suggests, this property is equivalent to the transitivity of the action of G on the arc-set of Γ.

We say that a transitive permutation group L is *graph-restrictive* if, for every arc-transitive graph (Γ, G) whose local group is permutation isomorphic to L, the order of the vertex-stabilizers is bounded from above by a constant $\mathbf{c}(L)$ depending only

ISSN: 2202-3518

on the group L. With this terminology, the famous Weiss Conjecture (posed in [10]) states that primitive groups are graph-restrictive. We refer to [6] for an extensive study of this notion.

Let (Γ, G) be an arc-transitive graph of valency d whose local group L is graph-restrictive. We choose a vertex α, and we consider a subgroup H of G generated by d distinct automorphisms, each one sending α to one of its d neighbours. A routine connectedness argument (see the proof of Theorem 6 for details) shows that H is transitive on the vertices of Γ. Therefore, by Frattini's Argument, $G = G_\alpha H$. Recall that $\mathbf{d}(G)$ denotes the minimal cardinality of a set of generators of G. We obtain that

$$\mathbf{d}(G) \leq |G_\alpha| + d \leq \mathbf{c}(L) + d$$

elements are sufficient to generate G. It follows that we can bound the minimal number of generators of G by a function of the valency of Γ alone. Indeed, let us define the function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ by

$$\mathbf{f}(d) = d + \max\{\mathbf{c}(L) \mid L \text{ graph-restrictive of degree } d\}.$$

Then, for every arc-transitive graph (Γ, G) of valency d whose local action is graph-restrictive,

$$\mathbf{d}(G) \leq \mathbf{f}(d).$$

More surprisingly, for every arc-transitive graph (Γ, G) of valency at most 4, the minimal number of generators of G is bounded by a constant regardless of the local group. The result is trivial for $d \in \{1, 2\}$. For $d \in \{3, 4\}$, some deeper concepts enter the picture. For every arc-transitive graph (Γ, G), there is a universal cover of the form $(\mathcal{T}_d, G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}})$, where α and β are two adjacent vertices of Γ, and $\mathcal{T}_d$ is the infinite tree of valency d. For every amalgamated product appearing in the universal cover for valency 3 and 4, an explicit presentation has been produced: see [2, 3, 5]. In Section 2, we will prove the following result.

Lemma 1. *Let (Γ, G) be an arc-transitive graph of valency $d \in \{3, 4\}$. Then*

$$\mathbf{d}(G) \leq 3,$$

and this bound is sharp.

One could dare to conjecture that there exists a function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ that takes the valency of the graph Γ as input, and returns an upper bound for $\mathbf{d}(G)$. The main contribution of this note is proving that such a function cannot exist.

Theorem 2. *There exists no function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ such that, for every arc-transitive graph (Γ, G) of valency d,*

$$\mathbf{d}(G) \leq \mathbf{f}(d).$$

Remark 3. To prove the veracity of Theorem 2, we will exhibit an infinite family $\mathcal{F}$ of pairs (Γ_h, G_h) such that the valency of the graphs is a constant (at least 8), while $\mathbf{d}(G_h)$ grows linearly with the exponent of the group. We would like to remark that, although the philosophies of the constructions are profoundly different, the graphs Γ_h carry an outstanding similarity with those built in [4, 9, 7] to prove that, for some imprimitive local groups of degree 6, the order of the vertex-stabilizers grows exponentially with the number of vertices of the graph.

We also observe that, in our construction, G is not the automorphism group of Γ. This prompts the following question.

Problem 4. Is there a function $\mathbf{f} : \mathbb{N} \to \mathbb{N}$ such that, if Γ is a connected arc-transitive graph of valency d, then

$$\mathbf{d}\,(\mathrm{Aut}(\Gamma)) \leq \mathbf{f}(d)?$$

Moreover, for our current and limited knowledge of arc-transitive graphs (Γ, G), having $\mathbf{d}(G)$ bounded appears to be quite common. Therefore, we ask the following.

Problem 5. Which assumptions on the arc-transitive graph (Γ, G) are needed to bound $\mathbf{d}(G)$ with a function of the valency (or the local group)?

To conclude, we give a bound on the order of the group G appearing in a vertex-transitive graph (Γ, G) depending on the valency d and the exponent of G. We underline that the exponent of G is not a local feature of the graph. For instance, if Γ is a cycle of odd length, then the local group is isomorphic to C_2, while the exponent of $\mathrm{Aut}(\Gamma)$ is twice the length of the cycle. (We denote the vertex-set of Γ by the symbol $V\Gamma$.)

Theorem 6. *There exists a function* $\mathbf{B} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ *such that, for every vertex-transitive graph* (Γ, G) *where the valency of* Γ *is* d*, and that the exponent of* G *is* e*,*

$$|V\Gamma| \leq \mathbf{B}(d, e) \quad \textit{and} \quad |G| \leq \mathbf{B}(d, e)!\,.$$

Finally, we point out that the function $\mathbf{B}$ appearing in Theorem 6 is the solution of the Burnside Restricted Problem (see [11, 12]). We also remark that the bound on the number of vertices is sharp. Indeed, let G be the largest finite group with $\mathbf{d}(G) = d$ and exponent e, and let S be a generating set of cardinality d. Then $\mathrm{Cay}(G, S)$ has precisely $\mathbf{B}(d, e)$ vertices.

2 Proof of Lemma 1

The bulk of the proof of Lemma 1 relies on the following observation.

Lemma 7. *Let* (Γ, G) *be an arc-transitive* d*-valent graph, and let* $\alpha \in V\Gamma$ *be a vertex. Then*

$$\mathbf{d}(G) \leq \mathbf{d}(G_\alpha) + 1.$$

Note that this bound is not sharp in general.

Proof. Let $\{\alpha,\beta\}$ be an edge of Γ and let $x \in G_{\{\alpha,\beta\}} \setminus G_{\alpha\beta}$ be an *edge-flip*, that is, an automorphism satisfying $\alpha^x = \beta$ and $\beta^x = \alpha$. (Examples of such elements are the generators y in the presentations of [3] and the generators a in [5]). We define two subgroups of G as

$$H := \langle G_\alpha, x\rangle \quad \text{and} \quad K := \langle G_\alpha, G_\beta\rangle.$$

It is well known that K defines either one or two orbits on $V\Gamma$, and, if they are distinct, α and β lie in distinct K-orbits. As x swaps α and β, we have that $K \le H$, and that H is transitive on $V\Gamma$. Since $G_\alpha \le H$, by Frattini's Argument, $G = H$. In particular, by construction of H,

$$\mathbf{d}(G) = \mathbf{d}(H) \le \mathbf{d}(G_\alpha) + 1. \qquad \square$$

Let us assume that (Γ, G) is an arc-transitive graph of valency 3. The five possible amalgam types for this case have been collected in [3]. We observe that the possibility for a vertex-stabilizer are

$$G_\alpha \in \{1,\, C_3, \mathrm{Sym}(3),\, D_6,\, \mathrm{Sym}(4),\, \mathrm{Sym}(4)\times C_2\}.$$

It is easy to check that all these groups are 2-generated. Hence, Lemma 7 concludes the proof in this case.

We turn to the scenario where the valency of Γ is 4. We need to consider three cases.

First, we suppose that the local group is dihedral. There are infinitely many amalgams whose local group is isomorphic to D_4, and these amalgams are classified in [2]. Using the notation from [2], we deduce that $G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}$ admits a generating set of the form $\{x, a_0, a_1, \ldots, a_{n-1}, y\}$, with $n \ge 2$. (Note that $\{x, a_0, a_1, \ldots, a_{\lceil (n-1)/2\rceil}\}$ is a minimal generating set for G_α, thus we cannot apply Lemma 7.) We also recall, from [2], that

$$\begin{aligned} a_i^x &= a_{n-1-i} \quad \text{for every } 0 \le i \le n-1,\\ a_i^y &= a_{n-i} \quad \text{for every } 1 \le i \le n-1. \end{aligned}$$

We compute, for every $0 \le i \le n-2$,

$$a_i^{xy} = a_{n-1-i}^y = a_{n-n+i+1} = a_{i+1}.$$

It follows that $\{x, a_0, y\}$ is a generating set for $G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}$, and hence $\mathbf{d}(G) \le 3$.

Now, we assume that the local group is not dihedral and that G is s-arc-transitive, for some $s \ge 1$. Without loss of generality, replacing s if necessary, we may also assume that G is not $(s+1)$-arc-transitive. If $s = 1$, then every vertex-stabilizer is isomorphic either to C_4 or to $C_2 \times C_2$. If $s \ge 2$, then the amalgams have been classified in [5]. If $s = 1$, or if $s \ge 2$ and

$$G_\alpha \in \{\mathrm{Alt}(4),\, \mathrm{Sym}(4),\, C_3 \times \mathrm{Alt}(4),\, \mathrm{Sym}(3)\times \mathrm{Sym}(4)\},$$

then G_α is 2-generated. In all cases under consideration, by Lemma 7, $\mathbf{d}(G) \le 3$, as desired.

To conclude, there are precisely four amalgams of index $(4, 2)$ left. To complete the proof for the upper bound, it is enough to manipulate their explicit presentations in [5] to identify a generating set of cardinality 3. There are two amalgams with G_α isomorphic to $C_3 \rtimes \mathrm{Sym}(3)$. In the first case, $\{x, t, ac\}$ is a generating set for $G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}$ in view of

$$a = acdcd^{-1}c = acacacac^{-1}ac = (ac)^3(ac)^t(ac),$$
$$c = a(ac), \quad y = x^t, \quad d = c^a.$$

In the second case, we find that $\{x, c, a\}$ generates $G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}$ as

$$t = a^2, \quad y = x^t, \quad d = c^a.$$

For the 4-arc-transitive case, we find that $\{t, c, a\}$ is a generating set for $G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}$ in view of

$$d = (c^t)^{-1}, \quad e = d^a,$$
$$x = (et)^{-4}, \quad y = x^a.$$

Meanwhile, for the 7-arc-transitive amalgams, $G_\alpha *_{G_{\alpha\beta}} G_{\{\alpha,\beta\}}$ can be generated by $\{h, p, a\}$, because

$$k = h^{-2}, \quad v = k^a k^{-1}, \quad q = (p^a)^{-1},$$
$$r = qq^h, \quad s = (r^a)^{-1},$$
$$t = (s^h)^{-1}pq^{-1}r^{-1}s^{-1}, \quad u = (t^a)^{-1}.$$

We have thus proved that a minimal generating set for G contains at most three elements. To prove that this bound is sharp it is sufficient to inspect the census of arc-transitive graphs of valency 3 and 4 (see [1, 8]): in doing so, we discover that most graphs have 3-generated automorphism groups. This completes the proof of Lemma 1. □

3 Proof of Theorem 2

Let h be a positive integer, and let p be a prime. We set

$$H := C_{p^h} \times C_{p^h} = \langle a, b \mid a^{p^h} = b^{p^h} = [a, b] = 1\rangle.$$

Let us consider the group algebra $\mathbb{F}_p[H]$ over the finite field with p elements. We define recursively the following chain of $\mathbb{F}_p[H]$-modules:

$$\gamma_0 := \mathbb{F}_p[H], \quad \text{and, for any } i \geq 1,$$

$$\gamma_i := [\gamma_{i-1}, H] = \langle v - vh \mid v \in \gamma_{i-1}, h \in H\rangle_{\mathbb{F}_p}.$$

Recall that the natural basis for the group algebra $\mathbb{F}_p[H]$ is

$$\left(a^i b^j \mid i, j \in \{0, 1, \ldots, p^h - 1\}\right).$$

For all $x, y \in \{0, 1, \ldots, p^h - 1\}$, we write $e_{xy} = (a-1)^x(b-1)^y \in \mathbb{F}_p[H]$. We claim that

$$\mathcal{B} = \left(e_{xy} \mid x, y \in \{0, 1, \ldots, p^h - 1\}\right)$$

is also a basis. As $\mathcal{B}$ and the natural basis have the same cardinality, to prove the claim we show that every element of the natural basis can be written as linear combinations of the elements of $\mathcal{B}$. First we prove, by induction on i, that

$$a^i = \sum_{x=0}^{i} \lambda_x e_{x0} \,. \tag{1}$$

Observe that $1 = e_{00} = a^0$ is an element of the natural basis and of $\mathcal{B}$. We can write

$$a^i = (a-1)\mathbf{p}_i(a) + 1,$$

where $\mathbf{p}_i$ is a polynomial in one variable with coefficients in $\mathbb{F}_p$ and degree $i-1$. By inductive hypothesis, for some suitable coefficients,

$$\mathbf{p}_i(a) = \sum_{x=0}^{i-1} \mu_x e_{x0}.$$

Hence,

$$a^i = (a-1)\mathbf{p}_i(a) + 1 = \sum_{x=0}^{i-1} \mu_x e_{(x+1)0} + e_{00} \,,$$

which proves Equation (1). Repeating the same argument for b^j, we can show that

$$b^j = \sum_{y=0}^{j} \lambda_y e_{0y} \,.$$

Therefore, for some suitable coefficients,

$$a^i b^j = \left(\sum_{x=0}^{i} \lambda_x e_{x0}\right)\left(\sum_{y=0}^{j} \lambda_y e_{0y}\right) = \sum_{x=0}^{i-1}\sum_{y=0}^{j-1} \lambda_x \lambda_y e_{xy},$$

which completes the proof of the claim.

For convenience, we set $e_{xp^h} = e_{p^h y} = 0$, for every $x, y \in \{0, 1, \ldots, p^h - 1\}$. Observe that

$$\begin{aligned} e_{xy} \cdot a &= (a-1)^x(b-1)^y \cdot a \\ &= (a-1)^x(1 + a - 1)(b-1)^y \\ &= (a-1)^x(b-1)^y + (a-1)^{x+1}(b-1)^y \\ &= e_{xy} + e_{(x+1)y}, \end{aligned}$$

and

$$\begin{aligned} e_{xy} \cdot b &= (a-1)^x (b-1)^y \cdot b \\ &= (a-1)^x (b-1)^y (1+b-1) \\ &= (a-1)^x (b-1)^y + (a-1)^x (b-1)^{y+1} \\ &= e_{xy} + e_{x(y+1)}. \end{aligned}$$

By a direct computation, we get that

$$\gamma_i = \langle e_{xy} \mid x+y \geq i \rangle_{\mathbb{F}_p}, \quad \text{and}$$

$$\gamma_i/\gamma_{i+1} = \langle e_{xy} + \gamma_{i+1} \mid x+y = i \rangle_{\mathbb{F}_p}.$$

Indeed, the formula holds for $\gamma_0 = \mathbb{F}_p[H]$, and by induction on i,

$$\begin{aligned} \gamma_i &= \langle e_{xy} \cdot a - e_{xy}, e_{xy} \cdot b - e_{xy} \mid x+y \geq i-1 \rangle_{\mathbb{F}_p} \\ &= \langle e_{(x+1)y}, e_{x(y+1)} \mid x+y+1 \geq i \rangle_{\mathbb{F}_p}. \end{aligned}$$

Recall that, for any $\mathbb{F}_p[H]$-module V, we denote by $\mathbf{d}_H(V)$ the minimal number of generators of V as an $\mathbb{F}_p[H]$-module. Since, by construction, γ_i/γ_{i+1} is a trivial section of $\mathbb{F}_p[H]$, we have that

$$\mathbf{d}_H\left(\gamma_i/\gamma_{i+1}\right) = \dim_{\mathbb{F}_p}\left(\gamma_i/\gamma_{i+1}\right) = \begin{cases} i+1 & \text{if } 0 \leq i \leq p^h - 1 \\ 2p^h - i - 1 & \text{if } p^h \leq i \leq 2(p^h-1) \\ 0 & \text{if } 2p^h - 1 \leq i. \end{cases} \tag{2}$$

We use this to compute the number of generators of $\gamma_{p^h} \rtimes H$. Indeed, we claim that

$$\mathbf{d}\left(\gamma_{p^h-1} \rtimes H\right) = p^h + 2\,. \tag{3}$$

First, we recall that, as $\gamma_{p^h-1} \rtimes H$ is a p-group,

$$\mathbf{d}\left(\gamma_{p^h-1} \rtimes H\right) = \dim_{\mathbb{F}_p}\left(\frac{\gamma_{p^h-1} \rtimes H}{\Phi(\gamma_{p^h-1} \rtimes H)}\right),$$

where $\Phi(\gamma_{p^h-1} \rtimes H)$ is the Frattini subgroup of $\gamma_{p^h-1} \rtimes H$. Second, we note that

$$\Phi(\gamma_{p^h-1} \rtimes H) = [\gamma_{p^h-1} \rtimes H, \gamma_{p^h-1} \rtimes H](\gamma_{p^h-1} \rtimes H)^p.$$

Since H is abelian, using standard commutator computations, we have

$$[\gamma_{p^h-1} \rtimes H, \gamma_{p^h-1} \rtimes H] = \gamma_{p^h}.$$

Moreover,

$$(\gamma_{p^h-1} \rtimes H)^p \geq H^p.$$

This shows that

$$\Phi(\gamma_{p^h-1} \rtimes H) \geq \gamma_{p^h} \rtimes H^p.$$

It is now time to recall that H acts trivially on the section $\gamma_{p^h-1}/\gamma_{p^h}$: this fact implies that the quotient

$$\frac{\gamma_{p^h-1} \rtimes H}{\gamma_{p^h} \rtimes H^p}$$

is abelian of exponent p. Therefore,

$$\Phi(\gamma_{p^h-1} \rtimes H) = \gamma_{p^h} \rtimes H^p,$$

and Equation (3) immediately follows from Equation (2).

After these preliminary considerations, we embark on the construction of the graph Γ_h. The group H acts regularly on the vertex-set of the Cayley graph defined by

$$\Delta := \mathrm{Cay}(H, \{a, a^{-1}, b, b^{-1}\}).$$

Recall that, for any two graphs Γ, Δ, the *wreath product of* Γ *by* Δ, denoted by $\Gamma \,\mathrm{wr}\, \Delta$, is the graph having vertex-set $V\Gamma \times V\Delta$, where (γ_1, δ_1) and (γ_2, δ_2) are adjacent if either $\delta_1 = \delta_2$ and $\{\gamma_1, \gamma_2\}$ is an edge of Γ, or $\{\delta_1, \delta_2\}$ is an edge of Δ. We define Γ_h as the wreath product of the empty graph on p vertices, $p\mathbf{K}_1$, by the Cayley graph Δ, that is,

$$\Gamma_h := p\mathbf{K}_1 \,\mathrm{wr}\, \Delta.$$

Note that, unless $p = 2$ and $h = 1$, Δ has valency 4, hence Γ_h has valency $4p$.

Observe that, as abstract groups, $C_p \mathrm{wr} H$ and $\mathbb{F}_p[H] \rtimes H$ are isomorphic. It follows that $\gamma_{p^h-1} \rtimes H$ is identified with a subgroup of $C_p \mathrm{wr} H$, which in turn is a subgroup of $\mathrm{Aut}(\Gamma_h)$. Moreover, $V\Gamma_h$ can be partitioned as

$$X := \{V(p\mathbf{K}_1) \times \{\delta\} \mid \delta \in V\Delta\}\,.$$

Note that X is γ_{p^h-1}-invariant, because the latter embeds in the base group of $C_p \mathrm{wr} H$. As γ_{p^h-1} is a nontrivial p-group, it must induce a transitive action on at least one part of X, while H permutes regularly the elements of X. It follows that $\gamma_{p^h-1} \rtimes H$ is transitive on the vertices of Γ_h, thus $(\Gamma_h, \gamma_{p^h-1} \rtimes H)$ is a vertex-transitive graph. On the other hand, since $\gamma_{p^h-1} \rtimes H$ preserves the lifting of the labels $\{a, a^{-1}, b, b^{-1}\}$ from the Cayley graph Δ, this action is not arc-transitive. In particular, the local group of $(\Gamma_h, \gamma_{p^h-1} \rtimes H)$ is intransitive with four distinct orbits.

To achieve the desired arc-transitivity, we extend the group H with some outer automorphisms. We consider the automorphisms φ and ψ of H defined on the generators by

$$\varphi:\ a \mapsto b,\ b \mapsto a, \quad \text{and} \quad \psi:\ a \mapsto a^{-1},\ b \mapsto b^{-1}.$$

Observe that φ and ψ are commuting involutions, thus $\langle \varphi, \psi \rangle$ is isomorphic to the Klein group. We extend the multiplication on $\mathbb{F}_p[H]$ by putting, for every $\varepsilon, \delta \in \mathbb{Z}_2$,

$$\left(\sum_{h \in H} \lambda_h h\right) \cdot \left(\varphi^{\varepsilon} \psi^{\delta}\right) = \sum_{h \in H} \lambda_h h^{\varphi^{\varepsilon} \psi^{\delta}}.$$

With this operation, $\mathbb{F}_p[H]$ is an $\mathbb{F}_p[H \rtimes \langle \varphi, \psi \rangle]$-module. Our putative subgroup of $\mathrm{Aut}(\Gamma_h)$ is

$$G_h := \gamma_{p^h-1} \rtimes (H \rtimes \langle \varphi, \psi \rangle).$$

Note that

$$\mathbb{F}_p[H] \rtimes H \geq \mathbb{F}_p[H] = \gamma_0 \geq \gamma_1 \geq \cdots \geq \gamma_{2(p^h-1)} \geq \gamma_{2p^h-1} = 1$$

is the central lower series of $\mathbb{F}_p[H] \rtimes H$, and hence, for all indices i, γ_i is a characteristic subgroup of $\mathbb{F}_p[H] \rtimes H$. It follows that γ_i is an $\mathbb{F}_p[H \rtimes \langle \varphi, \psi \rangle]$-submodule, and hence G_h is well-defined.

First, we give a lower bound on $\mathbf{d}(G_h)$, then we prove that (Γ_h, G_h) is an arc-transitive graph.

Let S be a generating set for G_h. The set $S \cup \{\varphi, \psi\}$ also generates G_h. By multiplying each element of S by a (possibly trivial) element of $\langle \varphi, \psi \rangle$, we can produce a new generating set for G_h of the form $T \cup \{\varphi, \psi\}$ where T is a subset of $\gamma_{p^h-1} \rtimes H$. We claim that

$$U := T^{\langle \varphi, \psi \rangle} \subseteq \gamma_{p^h-1} \rtimes H$$

is a generating set for $\gamma_{p^h-1} \rtimes H$. For every $g \in \gamma_{p^h-1} \rtimes H$, g can be written as a word in $T \cup \{\varphi, \psi\}$. Whenever φ appears in this word, we can move it to the right end of the word by conjugating by φ all the generators from its initial position to the end of the string. The same procedure can be applied to ψ. Once we have completed these operations, we find that g can be expressed as the product of two words: one in U and the other in $\{\varphi, \psi\}$. As $g \in \gamma_{p^h-1} \rtimes H$, the latter word must be trivial. This proves that we can express g as a word in U, and hence U generates $\gamma_{p^h-1} \rtimes H$. By construction, since $|\langle \varphi, \psi \rangle| = 4$,

$$|U| \leq 4|T| = 4|S|.$$

Hence, by choosing $|S|$ to be minimal,

$$\frac{1}{4}\mathbf{d}(\gamma_{p^h-1} \rtimes H) \leq \frac{1}{4}|U| \leq \mathbf{d}(G_h).$$

Therefore, using Equation (3),

$$\mathbf{d}(G_h) \geq \frac{p^h}{4}.$$

Let us go back to the Cayley graph Δ. Observe that $\langle \varphi, \psi \rangle$ is transitive on the connection set $\{a, a^{-1}, b, b^{-1}\}$ of Δ. This implies that $H \rtimes \langle \varphi, \psi \rangle$ is an arc-transitive subgroup of $\mathrm{Aut}(\Delta)$. Therefore,

$$G_h \leq C_p \,\mathrm{wr}(H \rtimes \langle \varphi, \psi \rangle) \leq \mathrm{Aut}(\Gamma_h).$$

Moreover, the local group of (Γ_h, G_h) transitively permutes the four orbits defined by the local group of $(\Gamma_h, \gamma_{p^h-1} \rtimes H)$. Hence, the pair (Γ_h, G_h) is an arc-transitive graph.

To wrap up, for a fixed prime p, the family

$$\mathcal{F}_p := \{(\Gamma_h, G_h) \mid h \geq 2\}$$

contains only graphs of valency $4p$; meanwhile

$$\lim_{h\to+\infty} \mathbf{d}(G_h) \geq \lim_{h\to+\infty} \frac{p^h}{4} = +\infty.$$

This family is a counterexample to the existence of a function $\mathbf{f}$ that, for every arc-transitive graph (Γ, G) of valency d, bounds $\mathbf{d}(G)$ in terms of d alone. Hence the proof of Theorem 2 is complete. □

4 Proof of Theorem 6

Let (Γ, G) be a vertex-transitive graph, and recall that d is the valency of Γ. We choose a vertex α of Γ, and, for any of its neighbours β, we consider an automorphism $g_\beta \in G$ such that

$$\alpha^{g_\beta} = \beta.$$

Recall that these elements exist by transitivity of G on $V\Gamma$. Hence, we can define the subgroup of G

$$H := \langle g_\beta \mid \beta \in \Gamma(\alpha)\rangle.$$

Now, we will present the connectedness argument mentioned in the introduction to prove that H is transitive on the vertex-set of Γ. Aiming for a contradiction, suppose that H is not transitive on $V\Gamma$. We choose a vertex γ at minimal distance from α which is not contained in the H-orbit of α. As Γ is connected, we can choose a vertex δ adjacent to γ such that

$$d_\Gamma(\alpha, \delta) + 1 = d_\Gamma(\alpha, \gamma).$$

By our choice of γ, there is an element $h \in H$ such that $\alpha^h = \delta$. Observe that

$$X := \left\{h^{-1} g_\beta h \mid \beta \in \Gamma(\alpha)\right\}$$

is a subset of H with the property that the image of δ under X is the neighbourhood of δ. In particular, hX contains an automorphism of H mapping α to γ. Thus, γ belongs to the H-orbit of α, a contradiction.

Let $\mathbf{B}(d, e)$ be the function solving the Burnside Restricted Problem for a finite group with d generators and exponent e. The existence of this function for all the choices of d and e was proved by Zel'manov in [11, 12]. Observe that, as H is a subgroup of G, and as the exponent of G is e, the exponent of H divides e. Thus, we find that the order of H is bounded from above by $\mathbf{B}(d, e)$. Moreover, since H is transitive on the vertex-set of Γ,

$$|V\Gamma| \leq |H| \leq \mathbf{B}(d, e).$$

This proves the first part of Theorem 6.

To complete the proof, it is enough to observe that G can be embedded into $\mathrm{Sym}(V\Gamma)$, which in turn embeds into $\mathrm{Sym}(\mathbf{B}(d,e))$. Therefore,

$$|G| \le \mathbf{B}(d,e)!,$$

as desired. □

Acknowledgements

The work in this paper is funded by the European Union – Next Generation EU, Missione 4 Componente 1 CUP B53D23009410006, PRIN 2022 - 2022PSTWLB – Group Theory and Applications. The authors are members of the GNSAGA INdAM research group and kindly acknowledge its support.

We are grateful to one of the referees for bringing the current proof of Lemma 7 to our attention during the evaluation process. Their insight helped simplify our earlier version, for which we extend our sincere thanks.

References

[1] M. D. E. Conder and P. Dobcsányi, Applications and adaptations of the low index subgroups procedure, *J. Combin. Math. Combin. Comput.* **40** (2002), 41–63.

[2] D. Ž. Djoković, A class of finite group-amalgams, *Proc. Amer. Math. Soc.* **80** (1980), 22–26.

[3] D. Ž. Djoković and G. L. Miller, Regular groups of automorphisms of cubic graphs, *J. Combin. Theory Ser. B* **29**(2) (1980), 195–230.

[4] A. Hujdurović, P. Potočnik and G. Verret, Three local actions in 6-valent arc-transitive graphs, *J. Graph Theory* **99**(2) (2022), 207–216.

[5] P. Potočnik, A list of 4-valent 2-arc-transitive graphs and finite faithful amalgams of index (4, 2), *European J. Combin.* **30**(5) (2009), 1323–1336.

[6] P. Potočnik, P. Spiga and G. Verret, On graph-restrictive permutation groups, *J. Combin. Theory Ser. B* **102**(3) (2012), 820–831.

[7] P. Potočnik, P Spiga, and G. Verret, On the nullspace of arc-transitive graphs over finite fields, *J. Algebraic Combin.* **36** (2012), 389–401.

[8] P. Potočnik, P. Spiga and G. Verret, Cubic vertex-transitive graphs on up to 1 280 vertices, *J. Symbolic Comput.* **50** (2013), 465–477.

[9] P. Potočnik, P. Spiga and G. Verret, On the order of arc-stabilizers in arc-transitive graphs with prescribed local group, *Trans. Amer. Math. Soc.* **366**(7) (2014), 3729–3745.

[10] R. Weiss, s-transitive graphs, *Colloq. Math. Soc. János Bolya* **25** (1978), 827–847.

[11] E. I. Zel'manov, Solution of the restricted Burnside problem for groups of odd exponent, *Math. USSR Izv.* **36** (1991), 41–60.

[12] E. I. Zel'manov, A solution of the restricted Burnside problem for 2-groups, *Math. USSR Sb.* **72**(2) (1992), 543–565.

(Received 23 May 2024; revised 4 June 2024, 30 Sep 2024)