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Rado functionals and applications

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Abstract

We study Rado functionals and the maximal Rado condition (first introduced by Barret, Lupini, and Moreira in 2021) in terms of the partition regularity of mixed systems of linear equations and inequalities. By strengthening the maximal Rado condition, we provide a sufficient condition for the partition regularity of polynomial equations over some infinite subsets of a given integral domain. By applying these results, we derive an extension of a previous result obtained by Di Nasso and Luperi Baglini concerning partition regular inhomogeneous polynomials in three variables and also conditions for the partition regularity of equations of the form $H(xz^{\rho}, y) = 0$, where ρ is a non-zero rational and $H \in \mathbb{Z}[x, y]$ is a homogeneous polynomial.

1 Introduction

Throughout this article, R denotes a commutative ring with unity; given any $S \subseteq R$, we denote by $S^{\times}$ the set $S \setminus \{0\}$. We assume the convention that $\mathbb{N}$ is the set of positive integers, i.e. $\mathbb{N} = \{1, 2, \dots\}$; for any $n \in \mathbb{N}$, $[n]$ denotes the set $\{1, \dots, n\}$. As usual, we denote the set of all polynomials in n variables over R by $R[x_1, \dots, x_n]$ and, given $m, n \in \mathbb{N}$, we let $\mathrm{Mat}_{m\times n}(R)$ be the set of all $m \times n$ matrices with entries

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in R. Whenever $A \in \mathrm{Mat}_{m\times n}(R)$ and $(a_1, \dots, a_m) \in R^m$, we omit the transposition sign and write $A(a_1, \dots, a_m)$ to mean $A(a_1, \dots, a_m)^{\mathrm{T}}$.

A long-studied problem in combinatorics is the partition regularity of systems of Diophantine equations.

Definition 1.1. Given $P_1, \dots, P_m \in R[x_1, \dots, x_n]$, we say that the system of equations

$$\sigma(x_1, \dots, x_n) := \begin{cases} P_1(x_1, \dots, x_n) & = & 0 \\ \vdots & \vdots & \vdots \\ P_m(x_1, \dots, x_n) & = & 0 \end{cases}$$

is partition regular over S if for every finite coloring[1] c of S there are c-monochromatic $a_1, \dots, a_n \in S$ (namely, all belonging to $c^{-1}[i]$ for some $i \le r$) satisfying $\sigma(a_1, \dots, a_n) = 0$.

This system is non-trivially partition regular over S if for every coloring c of S there are c-monochromatic $a_1, \dots, a_n \in S$ and distinct $i, j \in [n]$ such that $a_i \neq a_j$.

We say that $\sigma(x_1, \dots, x_n) = 0$ is infinitely partition regular over S if for every coloring c of S there are infinitely many c-monochromatic n-tuples $(a_1, \dots, a_n) \in S^n$ satisfying $\sigma(a_1, \dots, a_n) = 0$.

When $m = 1$, we will simply say that the polynomial P_1 is (infinitely) partition regular to mean that the equation $P_1(x_1, \dots, x_n) = 0$ is (infinitely) partition regular.

In 1933 in [20], Rado completely characterized which systems of linear equations are partition regular over $\mathbb{N}$ in terms of the so-called columns condition, which we formulate here in a more general version for R.

Definition 1.2. Let $A \in \mathrm{Mat}_{m\times n}(R)$ and $C_1, \dots, C_m$ be the columns of A; we say that A satisfies the columns condition if there is a partition $I_0, \dots, I_r$ of $[m]$ such that

1. $\sum_{i\in I_0} C_i = \vec{0}$; and
2. given any $u \in [r]$, $\sum_{i\in I_u} C_i \in \mathrm{span}_K\{C_j : j \in I_0 \cup \cdots \cup I_{u-1}\}$, where K is the field of fractions of R.

Homogeneous Rado's Theorem. *Given a matrix $A \in \mathrm{Mat}_{m\times n}(\mathbb{Q})$, the system $A\vec{t} = \vec{0}$ is infinitely partition regular over $\mathbb{N}$ if and only if A satisfies the columns condition.*

In [21], Rado proved the analogous result of the Homogeneous Rado's Theorem for subrings of $\mathbb{C}$. Also, in this article, Rado proved the characterization of all possible inhomogeneous linear systems that are partition regular, which reads as follows:

Inhomogeneous Rado's Theorem. *Given $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ and $\vec{b} \in \mathbb{Z}^m$, the system $A\vec{t} = \vec{b}$ is partition regular over $\mathbb{N}$ if and only if either*

[1] A finite coloring of S is a function $c : S \to \{1, \dots, r\}$ for some $r \in \mathbb{N}$. Since we only deal with finite colorings of sets in this work, from now on we refer to such functions simply as *colorings*.

1. *there is a constant solution $s \in \mathbb{N}$, i.e. $A(s, \ldots, s) = \vec{b}$; or*

2. *A satisfies the columns condition and there is a constant solution $s \in \mathbb{Z}$.*

We call *Rado system* any system of linear homogeneous equations that is partition regular. Recently, the following generalization of Rado's Theorem was proved:

Theorem 1.3. *[7, Theorem A] Let D be an infinite integral domain and $A \in \mathrm{Mat}_{m\times n}(D)$. Then the system $A\vec{t} = 0$ is infinitely partition regular over $D \setminus \{0\}$ if and only if A satisfies the columns condition.*

A study of non-trivial solutions to linear systems was done by Hindman and Leader in [15]. More on non-constant solutions or injective partition regularity can be found in [10]; we also recall that a linear partition regular system $A\vec{t} = 0$ over an infinite integral domain D will always be infinitely partition regular over $D \setminus \{0\}$ (see [5, Theorem 2.4] and Observation 2.11).

Several generalizations of Rado's Theorem were proved for commutative rings and infinite integral domains [5, 7]. Although the literature for linear systems is quite extensive and general, the one for the nonlinear case is scarce and mostly restricted to $\mathbb{Z}$. The articles [2, 4, 6, 8, 10, 12, 18, 19] contain the latest results regarding the partition regularity of nonlinear equations that we are aware of.

In this article we will build up upon results first proven by Barret, Lupini, and Moreira in [4] as generalizations of preliminary results proven by Di Nasso and Luperi Baglini in [10], to study Rado sets, Rado functionals and their implications for the partition regularity of equations. The case of polynomials in three variables will then be studied in more detail.

This paper is structured as follows: In Section 2 we recall the basic definitions of Rado partitions, sets, and functionals for a given polynomial $P \in R[x_1, \ldots, x_n]$, and we prove some implications and equivalences of these concepts in terms of the partition regularity of systems of linear equations and inequalities. In Section 3 we revise the maximal Rado condition and provide a strengthening of the Rado condition that is sufficient for the partition regularity over rings. Section 4 is devoted to applying Rado functionals to the case of inhomogeneous polynomials in three variables; in particular, we give a necessary and sufficient condition for the partition regularity of equations of the form $H(xz^\rho, y) = 0$, where $\rho \in \mathbb{Q}^\times$ and $H \in \mathbb{Z}[x, y]$ is a homogeneous polynomial.

2 Rado Functionals

2.1 Lower and Upper Rado Functionals

Building on some nonstandard characterizations first introduced in [10], in [4] Barret, Lupini, and Moreira introduced the notions of Rado sets and (upper and lower) Rado functionals and proved a necessary condition for the partition regularity of Diophantine equations, namely the maximal Rado condition (that we will discuss in detail in Section 3). In this section, we provide some explicit characterizations of Rado sets and functionals in terms of the partition regularity of mixed systems of linear equalities and inequalities. This will allow us to deduce necessary conditions on the structure of Rado sets.

We start by recalling the basic definitions from [4].

Definition 2.1. Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}$ be a linear map with coefficients $t_1, \dots, t_n \in \mathbb{N}$, i.e. $\varphi(a_1, \dots, a_n) = t_1 a_1 + \cdots + t_n a_n$. Given a coloring c of $\mathbb{N}$, we say that φ is c-monochromatic if its coefficients are c-monochromatic, i.e. if $c(t_1) = \cdots = c(t_n)$.

A multi-index is any element $\alpha = (\alpha(1), \dots, \alpha(n)) \in \mathbb{N}_0^n$ for some $n \in \mathbb{N}$. Let $\mathbb{N}_0^{<\omega} = \bigcup_{n\in\mathbb{N}} \mathbb{N}_0^n$. Given $\alpha \in \mathbb{N}_0^{<\omega}$, we let $\boldsymbol{x}^\alpha := x_1^{\alpha(1)} \cdot \dots \cdot x_n^{\alpha(n)}$, so that $|\alpha| = \alpha(1) + \cdots + \alpha(n)$ is the *degree* of $\boldsymbol{x}^\alpha$. Given any polynomial $P \in R[x_1, \dots, x_n]$, for every $\alpha \in \mathbb{N}_0^{<\omega}$ there is $c_\alpha \in R$ such that $P(\boldsymbol{x}) = \sum_\alpha c_\alpha \boldsymbol{x}^\alpha$ and the set $\mathrm{supp}(P) := \{\alpha \in \mathbb{N}_0^n : c_\alpha \neq 0\}$ (called the *support* of P) is finite. We say that $J \subseteq \mathrm{supp}(P)$ is *homogeneous* if for all $\alpha, \beta \in J$, $|\alpha| = |\beta|$; in particular, P is homogeneous if and only if $\mathrm{supp}(P)$ is. Given $i \in [n]$ and a subset $J \subseteq \mathrm{supp}(P)$, we say that the exponent of the variable x_i is *constant* in J if for all $\alpha, \beta \in J$ one has that $\alpha(i) = \beta(i)$.

Definition 2.2. Given a polynomial $P \in R[x_1, \dots, x_n]$, a coloring c of $\mathbb{N}$ and a c-monochromatic linear map $\varphi : \mathbb{Z}^n \to \mathbb{Z}$, let $M_0, \dots, M_l$ be an enumeration of $\varphi[\mathrm{supp}(P)]$; we say that a partition $J_0, \dots, J_l$ of $\mathrm{supp}(P)$ is determined by φ if for every $i \in [0, l]$, $J_i = \varphi^{-1}[\{M_i\}]$, i.e. J_i is the fiber $\{\alpha \in \mathrm{supp}(P) : \varphi(\alpha) = M_i\}$.

A partition $J_0, \dots, J_l$ for $\mathrm{supp}(P)$ is said to be a Rado partition if there are infinitely many c-monochromatic linear maps φ such that $J_0, \dots, J_l$ is determined by φ.

A Rado set over P is any $J \subseteq \mathrm{supp}(P)$ such that there are a Rado partition $(J_0, \dots, J_l)$ and $i \in [0, l]$ such that $J = J_i$.

When trying to find all possible Rado partitions of the support of P, the first question to answer is which subsets of $\mathrm{supp}(P)$ can be Rado sets. In one direction, a trivial characterization can be given in terms of partition regular systems.

Lemma 2.3. *Let $k \geq 2$ and $J = \{\alpha_1, \dots, \alpha_k\}$ be a Rado set for a polynomial $P \in R[x_1, \dots, x_n]$; then, given any $j \in [k]$, the matrix*

$$M_j(J) = \begin{pmatrix} \alpha_1(1) - \alpha_j(1) & \alpha_1(2) - \alpha_j(2) & \dots & \alpha_1(n) - \alpha_j(n) \\ \alpha_2(1) - \alpha_j(1) & \alpha_2(2) - \alpha_j(2) & \dots & \alpha_2(n) - \alpha_j(n) \\ \vdots & \vdots & \dots & \vdots \\ \alpha_{j-1}(1) - \alpha_j(1) & \alpha_{j-1}(2) - \alpha_j(2) & \dots & \alpha_{j-1}(n) - \alpha_j(n) \\ \alpha_{j+1}(1) - \alpha_j(1) & \alpha_{j+1}(2) - \alpha_j(2) & \dots & \alpha_{j+1}(n) - \alpha_j(n) \\ \vdots & \vdots & \dots & \vdots \\ \alpha_k(1) - \alpha_j(1) & \alpha_k(2) - \alpha_j(2) & \dots & \alpha_k(n) - \alpha_j(n) \end{pmatrix}$$

satisfies the columns condition.

Proof. Giving a colouring c for $\mathbb{N}$, there must be (infinitely many) c-monochromatic positive linear maps $\varphi : \mathbb{Z}^n \to \mathbb{Z}$ such that, for each $u, v \in [k]$, one has $\varphi(\alpha_u) = \varphi(\alpha_v)$; if $t_1, \ldots, t_n \in \mathbb{N}$ are the coefficients of φ, we have that

$$\big(\alpha_u(1) - \alpha_v(1)\big)t_1 + \cdots + \big(\alpha_u(n) - \alpha_v(n)\big)t_n = 0.$$

Hence, picking any $j \in [k]$, we have that $M_j(J)(t_1, \ldots, t_n) = \vec{0}$ and, by definition, $t_1, \ldots, t_n$ are c-monochromatic. Thus, the system $M_j(J)(t_1, \ldots, t_n) = \vec{0}$ is partition regular and, by the Homogeneous Rado's Theorem, one deduces that $M_j(J)$ satisfies the columns condition. $\square$

Let us observe that the columns condition is a property that is preserved under Gaussian operations; as such if $J = \{\alpha_1, \ldots, \alpha_k\}$ is a Rado set for $P \in \mathbb{Z}[x_1, \ldots, x_n]$ and $M_1(J)$ satisfies the columns condition, then for each $j \in [2, k]$, the matrix $M_j(J)$ also satisfies the columns condition. Hence, it is enough to work with the matrix $M(J) := M_1(J)$.

Lemma 2.3 cannot be reversed, in general, as being a Rado set is a condition that involves the whole support of P: for example, if $P(x, y, z) = x^4 + y^4z^2 + x^2y^2z$, the set $\{(4, 0, 0), (0, 4, 2)\}$ satisfies the conclusion of Lemma 2.3, but it is not a Rado set as any linear map $\varphi : \mathbb{N}_0^3 \to \mathbb{N}_0$ with $\varphi((4, 0, 0)) = \varphi((0, 4, 2)) = c$ necessarily gives also $\varphi((2, 2, 1)) = c$.

However, the following result shows that Lemma 2.3 can be reversed if we add a maximality hypothesis on J.

Proposition 2.4. *Let $J \subseteq \operatorname{supp}(P)$ be such that $M(J)$ satisfies the columns condition but, for any $\alpha \in \operatorname{supp}(P) \setminus J$, $M(J \cup \{\alpha\})$ does not satisfy the columns condition. Then, J is a Rado set.*

Proof. Let c be a given coloring of $\mathbb{N}_0$. Define

$$T(c) = \{\vec{t} \in \ker M(J) : \vec{t} \text{ is } c\text{-monochromatic}\}.$$

Let $K = \operatorname{supp}(P) \setminus J$ and let $\mathfrak{P}$ be the collection of all possible partitions of K. Given $\vec{t} \in T(c)$, let $M_1, \ldots, M_l$ be the possible values of the map $\varphi_{\vec{t}}$ defined as $\alpha \mapsto \alpha \cdot \vec{t}$ ($\alpha \in \mathbb{Z}^n$) applied to K; define $\mathcal{P}_{\vec{t}}$ as the partition of J determined by this map, i.e. $\mathcal{P}_{\vec{t}} = \{K_1, \ldots, K_l\}$, where for each $i \in [l]$, $K_i = \{\beta \in K : \beta \cdot \vec{t} = M_i\}$. Since $T(c)$ is infinite and $\mathfrak{P}$ is finite, the set

$$S(c) = \big\{\mathcal{P} \in \mathfrak{P} : \{\vec{t} \in T(c) : \mathcal{P} = \mathcal{P}_{\vec{t}}\} \text{ is infinite}\big\}$$

is not empty.

We claim that

$$\mathcal{S} = \{S(c) : c \text{ is a coloring of } \mathbb{N}_0\}$$

has the finite intersection property. Indeed, for each $v \in \mathbb{N}$ and $i \in [v]$, let $c_i : \mathbb{N}_0 \to \{1, \ldots, m_i\}$ be a coloring of $\mathbb{N}_0$. Let $p_1 < \cdots < p_v$ be prime numbers and

$m = \prod_{i=1}^{v} p_i^{m_i}$; define $c : \mathbb{N}_0 \to \{1, \ldots, m\}$ as

$$c(a) = \prod_{i=1}^{v} p_i^{c_i(a)}.$$

Let us observe that $T(c) = T(c_1) \cap \cdots \cap T(c_v)$; this implies that $S(c) \subseteq S(c_1) \cap \cdots \cap S(c_v)$ and settles the claim. As $\mathcal{S}$ is finite, there must be a $\mathcal{P} \in \bigcap \mathcal{S}$.

Finally, if c is any coloring of $\mathbb{N}_0$, there must be infinitely many monochromatic $\vec{t}$ such that $\mathcal{P} = \mathcal{P}_{\vec{t}}$. By construction, for any distinct $K_1, K_2 \in \mathcal{P}$, $\alpha \in K_1$ and $\beta \in K_2$ we must have $\varphi_{\vec{t}}(\alpha) \neq \varphi_{\vec{t}}(\beta)$. Moreover, if $\alpha \in K$ and $\beta \in J$ are such that $\varphi_{\vec{t}}(\alpha) = \varphi_{\vec{t}}(\beta)$, then one can prove that $M(J \cup \{\alpha\})$ satisfies the columns condition, which contradicts the hypothesis. Consequently, $\{J\} \cup \mathcal{P}$ is a Rado Partition for $\mathrm{supp}(P)$, which makes J a Rado set. $\square$

Rado partitions are used to introduce upper and lower Rado functionals for a polynomial.

Definition 2.5. Let $P \in R[x_1, \ldots, x_n]$ and $m \in \mathbb{N}_0$. Given integers $m \leq l$ and $d_1, \ldots, d_m \in \mathbb{N}$, a tuple $(J_0, \ldots, J_l, d_1, \ldots, d_m)$ is said to be a lower Rado functional of order m for P if for all $r \in \mathbb{N}$ and for all colorings c of $\mathbb{N}_0$ there are infinitely many c-monochromatic positive linear maps φ such that

1. $(J_0, \ldots, J_l)$ is the partition determined by φ; and
2. if $M_0 < \cdots < M_l$ is the enumeration of $\varphi[\mathrm{supp}(P)]$ such that $J_i = \varphi^{-1}[M_i]$, then
   (a) for each $i \in [m]$, $M_i - M_0 = d_i$; and
   (b) if $m < l$, then[2] $M_{m+1} - M_m \geq r$.

Given $d_0, \ldots, d_{m-1} \in \mathbb{N}$, we say that $(J_0, \ldots, J_l, d_0, \ldots, d_{m-1})$ is an upper Rado functional of order m for P if for all $r \in \mathbb{N}$ and for all colorings c of $\mathbb{N}_0$ there are infinitely many c-monochromatic positive linear maps φ such that

1. $(J_0, \ldots, J_l)$ is the partition determined by φ; and
2. if $M_l < \cdots < M_0$ is the enumeration of $\varphi[\mathrm{supp}(P)]$ such that $J_i = \varphi^{-1}[M_i]$, then
   (a) for each $i \in [0, m-1]$, $M_i - M_m = d_i$; and
   (b) if $m < l$, then[3] $M_m - M_{m+1} \geq r$.

[2]Note that if $m = l$, this condition is omitted.
[3]See footnote 2.

Let us fix a notation that will help to deal with Rado functionals: given an upper Rado functional $\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{m-1})$ for $P \in R[x_1, \dots, x_n]$ and $i \in [0, l]$, enumerate J_i as $J_i = \{\alpha_{i,1}, \dots, \alpha_{i,k_i}\}$. We let

$$\widehat{N}_{\mathcal{J}} = \begin{pmatrix} \alpha_{0,1} - \alpha_{m,1} \\ \vdots \\ \alpha_{m-1,1} - \alpha_{m,1} \end{pmatrix} \text{ and } \widehat{A}_{\mathcal{J}} = \begin{pmatrix} M(J_0) \\ \vdots \\ M(J_l) \\ \widehat{N}_J \end{pmatrix}, \tag{2.1}$$

where the matrices $M(J_i)$ have been defined in Lemma 2.3. By the definition above, $\widehat{A}_{\mathcal{J}}$ has $u = k_0 + \cdots + k_l - l + m - 1$ lines. Define $\widehat{b} \in \mathbb{Z}^u$ as the vector whose first $k_0 + \cdots + k_l - l - 1$ coordinates are 0 and the remaining m coordinates are $d_0, \dots, d_{m-1}$ in this order.

Similarly, if $\mathcal{J} = (J_0, \dots, J_l, d_1, \dots, d_m)$ is a lower Rado functional, we define the associated matrices

$$\check{N}_{\mathcal{J}} = \begin{pmatrix} \alpha_{1,1} - \alpha_{0,1} \\ \vdots \\ \alpha_{m,1} - \alpha_{0,1} \end{pmatrix} \text{ and } \check{A}_{\mathcal{J}} = \begin{pmatrix} M(J_0) \\ \vdots \\ M(J_l) \\ \check{N}_J \end{pmatrix},$$

and $\check{b} \in \mathbb{Z}^u$ as the vector whose first $k_0 + \cdots + k_l - l - 1$ coordinates are 0 and the remaining m coordinates are $d_1, \dots, d_m$ in this order.

Lemma 2.3 can be easily generalized to a stronger result that characterizes upper Rado functionals in terms of mixed systems of linear equalities and inequalities. To this end, we need to introduce a definition.

Definition 2.6. Let $\vec{b} \in \mathbb{Z}^n$. We say that $\vec{b} \cdot \vec{t} = \infty$ is partition regular over $\mathbb{N}$ if and only if for all coloring c of $\mathbb{N}$ and all $l \in \mathbb{N}$ there exists a c-monochromatic $\vec{t} \in \mathbb{N}^n$ such that $\vec{b} \cdot \vec{t} > l$.

It is easy to see that, by the definition, if $\vec{b} \cdot \vec{t} = \infty$ is partition regular over $\mathbb{N}$, then it is infinitely partition regular over $\mathbb{N}$, in the sense that for all coloring c of $\mathbb{N}$ and all $l \in \mathbb{N}$ there exist infinitely many c-monochromatic $\vec{t} \in \mathbb{N}^n$ such that $\vec{b} \cdot \vec{t} > l$. Upper Rado functionals (and lower Rado functionals) can be easily characterized as follows:

Theorem 2.7. *Let $P \in R[x_1, \dots, x_m]$ and $d_0 > d_1 > \cdots > d_{m-1}$ be natural numbers. Then, the tuple $\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{m-1})$ is an upper Rado functional for P if* *and only if there are* $e_{m+1}, \ldots, e_l \in \mathbb{N} \cup \{\infty\}$ *such that the system*

$$\begin{cases} \widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b} \\ (\alpha_{m,1} - \alpha_{m+1,1}) \cdot \vec{t} & = & \infty \\ \vdots & \vdots & \vdots \\ (\alpha_{m,1} - \alpha_{l,1}) \cdot \vec{t} & = & \infty \\ (\alpha_{m+1,1} - \alpha_{m+2,1}) \cdot \vec{t} & = & e_{m+1} \\ (\alpha_{m+2,1} - \alpha_{m+3,1}) \cdot \vec{t} & = & e_{m+2} \\ \vdots & \vdots & \vdots \\ (\alpha_{l-1,1} - \alpha_{l,1}) \cdot \vec{t} & = & e_l \end{cases} \tag{2.2}$$

is infinitely partition regular over $\mathbb{N}$*. Similarly, given natural numbers* $d_1 < \cdots < d_l$*, the tuple* $\mathcal{J} = (J_0, \ldots, J_l, d_1, \ldots, d_m)$ *is a lower Rado functional for* P *if and only if there are* $e_{m+1}, \ldots, e_l \in \mathbb{N} \cup \{\infty\}$ *such that the system*

$$\begin{cases} \widecheck{A}_{\mathcal{J}}\vec{t} = \widecheck{b} \\ (\alpha_{m+1,1} - \alpha_{m,1}) \cdot \vec{t} & = & \infty \\ \vdots & \vdots & \vdots \\ (\alpha_{l,1} - \alpha_{m,1}) \cdot \vec{t} & = & \infty \\ (\alpha_{m+2,1} - \alpha_{m+3,1}) \cdot \vec{t} & = & e_{m+2} \\ \vdots & \vdots & \vdots \\ (\alpha_{l-1,1} - \alpha_{l,1}) \cdot \vec{t} & = & e_l \end{cases} \tag{2.3}$$

is infinitely partition regular over $\mathbb{N}$*.*

Proof. The proof is just a modification of that of Lemma 2.3. □

Observation 2.8. Although Theorem 2.7 fully characterizes upper (and lower) Rado functionals $\mathcal{J} = (J_0, \ldots, J_l, d_0, \ldots, d_{m-1})$, for most of our later applications we will not need to use the partition regular relations of the type $(\alpha - \beta) \cdot \vec{t} = e_i$ for $i \in [m+1, l]$, $\alpha \in J_i$, $\beta \in J_{i+1}$, and $e_i \in \mathbb{N} \cup \{\infty\}$. In fact, the information that we will need is that the Rado sets $J_0, \ldots, J_m$ are at a given finite distance from each other, computed by the $d_i's$, and that the other Rado sets are at an infinite distance from J_0; the relative distances between the Rado sets $J_{m+1}, \ldots, J_l$ will not be used anywhere. Hence, all the information that we will use will actually be given by the infinite partition regularity of a simplification of the system (2.2), namely the system

$$\begin{cases} \widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b} \\ (\alpha_{m+1,1} - \alpha_{m,1}) \cdot \vec{t} & = & \infty \\ \vdots & \vdots & \vdots \\ (\alpha_{l,1} - \alpha_{m,1}) \cdot \vec{t} & = & \infty \end{cases}$$

As shown, Rado functionals are connected with partition regular systems of equations over $\mathbb{N}$, which are intertwined with the study of ultrafilters and their algebra.

Although we will make limited use of ultrafilters in this paper, we need to recall a few very basic facts and definitions; for an introduction to the basic theory of ultrafilters and its applications in Ramsey theory, we refer to the monograph [16]. Let us start with a general definition.

Definition 2.9. [16, Definition 3.10] Let $\mathcal{C}$ a collection of sets; we say that $\mathcal{C}$ is partition regular (also called weakly partition regular) if given any coloring c of $\bigcup \mathcal{C}$, one can find a c-monochromatic $A \in \mathcal{C}$; i.e. for all $a, a' \in A$ one has $c(a) = c(a')$.

For instance, a system of polynomial equations over R is partition regular over $S \subseteq R$ if and only if the collection of all subsets of S that contain solutions of the system is a partition regular collection; the analogous result applies to systems of inequalities, the partition regular relation of Definition 2.6 or any mixed systems with such binary relations.

Theorem 2.10. *[16, Theorem 3.11] A collection $\mathcal{C}$ of subsets of a set S is partition regular if and only if for all $A \in \mathcal{C}$ there is an ultrafilter*

$$\mathcal{U} \subseteq \{B \subseteq S : \exists C \in \mathcal{C} \text{ such that } C \subseteq B\}$$

Such an ultrafilter is said to witness (or is a witness for) the partition regularity of $\mathcal{C}$.

Hence, for instance, a system of Diophantine equations is partition regular over $\mathbb{N}$ if and only if there is an ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ such that any $A \in \mathcal{U}$ contains a solution to the system in question. The analogous result applies to systems of inequalities, the partition regular relation of Definition 2.6 or any mixed systems with such binary relations. Clearly, the collection $\mathcal{C}$, as in Theorem 2.10, is infinitely partition regular if and only if its partition regularity is witnessed by a free ultrafilter.

Observation 2.11. Let A be a matrix with rational entries. The Homogeneous Rado's Theorem is usually stated as an equivalence between the columns condition for the matrix A and the partition regularity of the linear system $A\vec{t} = 0$ over $\mathbb{N}$. By [10, Theorem 2.4] and the fact that $A\vec{t} = 0$ is a homogeneous system if $A\vec{t} = 0$ is partition regular over $\mathbb{N}$, then any ultrafilter in the minimal bilateral ideal of $(\beta\mathbb{N}, \cdot)$ witnesses its partition regularity; in particular, we get the existence of free ultrafilters witnesses for such system. This shows that, for homogeneous systems of equations over $\mathbb{N}$, the partition regularity is thus equivalent to the infinite partition regularity. The same remark works mutatis mutandis for the partition regularity of linear homogeneous systems over infinite integral domains (see Theorem 1.3).

So, given any $A \in \mathrm{Mat}_{m\times n}(\mathbb{Q})$, the system $Ax = 0$ is partition regular over $\mathbb{N}$ if and only if it is infinitely partition regular; nevertheless, the same does not apply for inhomogeneous linear systems, as the equation $x + y + z = 3$ is partition regular over $\mathbb{N}$ but only admits one monochromatic solution, namely $x = y = z = 1$. The following result is a simple consequence of the Inhomogeneous Rado's Theorem and characterizes infinitely partition regular inhomogeneous linear systems; in later sections, we apply the following result to the study of Rado functionals.

Lemma 2.12. *Given $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$ and $\vec{d} \in \mathbb{Z}^m \setminus \{0\}$, suppose that the system $A\vec{t} = \vec{d}$ is partition regular but A does not satisfy the columns condition. Then the system $A\vec{t} = \vec{d}$ is not infinitely partition regular.*

Conversely, if A satisfies the columns condition and there is a constant solution $s \in \mathbb{Z}$ to the system $A\vec{t} = \vec{d}$, then the system $A\vec{t} = \vec{d}$ is infinitely partition regular. In particular, for $\vec{b}_1, \dots, \vec{b}_k \in \mathbb{Z}^n$, if the system

$$\begin{cases} A\vec{t} & = & \vec{d} \\ \vec{b}_1 \cdot \vec{t} & = & \infty \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} & = & \infty \end{cases} \tag{2.4}$$

is partition regular over $\mathbb{N}$, then it is infinitely partition regular over $\mathbb{N}$.

Proof. Suppose that $A\vec{t} = \vec{d}$ is partition regular but A does not satisfy the columns condition. Then, the Inhomogeneous Rado's Theorem proves that the system has a constant solution $s \in \mathbb{N}$ and the Homogeneous Rado's Theorem proves that there is a coloring $c : \mathbb{N} \to [k]$ such that any given solution $a_1, \dots, a_n$ to the system $A\vec{t} = 0$ cannot be c-monochromatic. Define the coloring $\chi : \mathbb{N} \to [k+1]$ as

$$\chi(x) = \begin{cases} c(x-s), & \text{if } x > s; \text{ or} \\ k+1, & \text{otherwise.} \end{cases}$$

Suppose that $a_1, \dots, a_n$ is a χ-monochromatic solution to $A\vec{t} = \vec{d}$ of color $j < k+1$, so that it satisfies $a_i > s$ for all $i \in [n]$; then $a_1 - s, \dots, a_n - s$ is a c-monochromatic solution to the system $A\vec{t} = 0$, and this is absurd. Therefore any χ-monochromatic solution $a_1, \dots, a_n$ to $A\vec{t} = \vec{d}$ must be $(k+1)$-colored, so it must satisfy $a_i \in [s]$ for all $i \in [n]$, which shows that such a system cannot be infinitely partition regular.

Now assume that A satisfies the columns condition and that there is a constant solution $s \in \mathbb{Z}$ to the system $A\vec{t} = \vec{d}$. We define χ as above. The Homogeneous Rado's Theorem proves (see Observation 2.11) that there are infinitely many χ-monochromatic solutions $a_1, \dots, a_n \in \mathbb{N}$ of $A\vec{t} = 0$ such that $a_i > |s|$ for all $i \in [k]$. Since $a_1 - s, \dots, a_n - s$ is a c-monochromatic solution to $A\vec{t} = \vec{d}$, one has that this inhomogeneous system is infinitely partition regular over $\mathbb{N}$.

Finally, as we already observed, the partition regularity of inequalities of the type $\vec{b} \cdot \vec{t} = \infty$ implies that such inequalities must be infinitely partition regular. Thus, if the system (2.4) is partition regular over $\mathbb{N}$, it must be infinitely partition regular. □

Theorem 2.7 has a few interesting consequences that force strict conditions on Rado partitions. In all the results below, we keep the same notations introduced above. We omit the consequences for the lower Rado functionals, as they are stated and derived in a totally similar fashion. Through the rest of this section, $P \in R[x_1, \dots, x_m]$ and $\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{m-1})$ is an upper Rado functional for P.

Corollary 2.13. *If $\mathcal{J}$ is an upper Rado functional for P, then the system $\widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b}$ admits a constant solution $s \in \mathbb{Z}$, and the matrix $\widehat{A}_{\mathcal{J}}$ satisfies the columns condition.*

Proof. By Theorem 2.7, the system $\widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b}$ admits infinitely many monochromatic solutions hence, by Lemma 2.12, it has a constant solution $s \in \mathbb{Z}$ and $\widehat{A}_{\mathcal{J}}$ must satisfy the columns condition. □

Theorem 2.7 has particularly restrictive consequences when $m \geq 1$ in the Rado functionals:

Corollary 2.14. *If $m \geq 1$, the sets $J_0, \ldots, J_l$ are homogeneous.*

Proof. By Corollary 2.13, there is $s \in \mathbb{Z}$ so that $\widehat{A}_{\mathcal{J}}(s, \ldots, s) = \widehat{b}$. As $m \geq 1$, $\widehat{b} \neq 0$, which implies that $s \neq 0$. Consequently, for each $i \in \{0, \ldots, l\}$, $M(J_i)(s, \ldots, s) = 0$. By the definition of $M(J_i)$, for each $j \in \{2, \ldots, k_i\}$

$$\begin{aligned} 0 =& \big(\alpha_{i,j}(1) - \alpha_{i,1}(1)\big)s + \big(\alpha_{i,j}(2) - \alpha_{i,1}(2)\big)s + \cdots + \big(\alpha_{i,j}(n) - \alpha_{i,1}(n)\big)s \\ =& \big(|\alpha_{i,j}| - |\alpha_{i,1}|\big)s. \end{aligned}$$

Since $s \neq 0$, it must be $|\alpha_{i,j}| - |\alpha_{i,1}| = 0$, namely each J_i is homogeneous. □

Corollary 2.15. *If $m \geq 1$, there exists $s \in \mathbb{Z} \setminus \{0\}$ such that for all $i \in \{0, \ldots, m-1\}$, $\beta_i \in J_i$ and $\alpha \in J_m$*

$$|\beta_i| = |\alpha| + \frac{d_i}{s}.$$

Proof. By Corollary 2.13, there is a solution $s \in \mathbb{Z} \setminus \{0\}$ to the system $\widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b}$. By the definition of the matrix $\widehat{N}_{\mathcal{J}}$, for all $i \in \{1, \ldots, m-1\}$,

$$d_i = \sum_{k=1}^{n} \big(\alpha_{i,1}(k) - \alpha_{m,1}(k)\big)s = \big(|\alpha_{i,1}| - |\alpha_{m,1}|\big)s. \tag{2.5}$$

Hence, we have that $s \neq 0$. For all $i \in [m-1]$, by Equation (2.5), we must have $|\alpha_{i,1}| = |\alpha_{m,1}| + \frac{d_i}{s}$, which concludes the proof. □

Corollary 2.16. *If $P \in R[x_1, \ldots, x_n]$ is homogeneous then any upper Rado functional for P must have order $m = 0$.*

Proof. We proceed by contradiction: Suppose that $m \geq 1$ and let

$$\mathcal{J} = (J_0, \ldots, J_l, d_0, \ldots, d_{m-1})$$

be an upper Rado functional for P of order m. By Corollary 2.14, the Rado sets $J_0, \ldots, J_l$ are homogeneous; for each $i \in [l]$, let L_i be the degree of any multi-index in J_i. Since P is homogeneous, $L_1 = \cdots = L_l$. Now, by Corollary 2.15, there exists an $s \in \mathbb{Z} \setminus \{0\}$ such that $L_i = L_m + \frac{d_i}{s}$; this is absurd since each d_i is positive. □

The above corollaries show that having an upper Rado functional with $m \geq 1$ forces very restrictive conditions on the Rado partition, both on its Rado sets and the increments in the functional.

To conclude this section, we want to characterize the partition regularity of systems of the form (2.2). When $\vec{b}$ is null and "$= \infty$" is substituted by "> 0", the partition regularity of such systems has been settled by Hindman and Leader in [14]:

Theorem 2.17. *[14, Theorem 2] Let A be an $m \times n$ matrix of rational entries and for each $j \in [k]$, let $\vec{b_j} = (b_{j1}, \ldots, b_{jn})$ be a vector with rational entries. Then the following are equivalent:*

1. *the system*
$$\begin{cases} A\vec{t} & = & 0 \\ \vec{b}_1 \cdot \vec{t} & > & 0 \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} & > & 0 \end{cases}$$
is partition regular over $\mathbb{N}$;

2. *there are positive rationals $q_1, \ldots, q_k$ such that the system of equations*
$$\begin{cases} A\vec{t} & = & 0 \\ \vec{b}_1 \cdot \vec{t} - q_1 z_1 & = & 0 \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} - q_d z_k & = & 0 \end{cases}$$
on the variables $\vec{t} = (t_1, \ldots, t_n)$, $z_1, \ldots, z_k$ is partition regular over $\mathbb{N}$.

However, to much of our surprise, and at the best of our knowledge, the general partition regularity of systems like (2.2) has not been characterized yet in the literature. Therefore, we provide such a characterization below. Firstly, we show that using "$= \infty$" and "$> \infty$" are equivalent when it comes to partition regularity.

Lemma 2.18. *Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{Q})$, $\vec{b}_1, \ldots, \vec{b}_k \in \mathbb{Z}^n$. Then, the following are equivalent*

1. *the system*
$$\begin{cases} A\vec{t} & = & 0 \\ \vec{b}_1 \cdot \vec{t} & = & \infty \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} & = & \infty \end{cases}$$
is partition regular over $\mathbb{N}$;

2. *the system*

$$\begin{cases} A\vec{t} & = & 0 \\ \vec{b}_1 \cdot \vec{t} & > & 0 \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} & > & 0 \end{cases}$$

is partition regular over $\mathbb{N}$.

Proof. That (1) implies (2) is immediate. Conversely, suppose that (2) holds. By Theorem 2.17 there are rationals $q_1, \ldots, q_k > 0$ such that the system

$$\begin{cases} A\vec{t} & = & 0 \\ \vec{b}_1 \cdot \vec{t} - q_1 z_1 & = & 0 \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} - q_k z_k & = & 0 \end{cases}$$

is partition regular over $\mathbb{N}$ on the variables $\vec{t}$ and $z_1, \ldots, z_k$. Since this system is homogeneous, [10, Corollary 2.5] implies the existence of a free ultrafilter $\mathcal{U} \in \beta\mathbb{N}$ such that every set $A \in \mathcal{U}$ contains a solution to this system. Let $r \in \mathbb{N}$ and pick a $d \in \mathbb{N}$ such that $d > \max\{rq_1^{-1}, \ldots, rq_k^{-1}\}$; then the set $I = [d, +\infty[$ is an element of $\mathcal{U}$ and thus there are $t_1, \ldots, t_n, z_1, \ldots, z_k \in I$ such that $A\vec{t} = 0$ and for each $i \in [k]$, $\vec{b}_i \cdot \vec{t} - q_i z_i = 0$. This implies that $\vec{b}_i \cdot \vec{t} > r$. □

We can now characterize the partition regularity of mixed inhomogeneous systems of linear equations and inequalities.

Theorem 2.19. *Let $A \in \mathrm{Mat}_{m\times n}(\mathbb{Z})$, $\vec{b}_1, \ldots, \vec{b}_k \in \mathbb{Z}^n$, and a non-zero $\vec{d} \in \mathbb{Z}^m$. Then, the following are equivalent*

1. *The system*

$$\begin{cases} A\vec{t} & = & \vec{d} \\ \vec{b}_1 \cdot \vec{t} & = & \infty \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} & = & \infty \end{cases} \tag{2.6}$$

is partition regular over $\mathbb{N}$;

2. *there is a constant $s \in \mathbb{Z}$ such that $A(s, \ldots, s) = \vec{d}$ and the system*

$$\begin{cases} A\vec{t} & = & 0 \\ \vec{b}_1 \cdot \vec{t} & = & \infty \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} & = & \infty \end{cases} \tag{2.7}$$

is partition regular over $\mathbb{N}$;

3. *there is a constant $s \in \mathbb{Z}$ such that $A(s, \ldots, s) = \vec{d}$ and there are positive $q_1, \ldots, q_k \in \mathbb{Q}_{>0}$ such that the system*

$$\begin{cases} A\vec{t} & = & 0 \\ \vec{b}_1 \cdot \vec{t} - q_1 z_1 & = & 0 \\ \vdots & \vdots & \vdots \\ \vec{b}_k \cdot \vec{t} - q_k z_k & = & 0 \end{cases}$$

is partition regular over $\mathbb{N}$.

Proof. Let us show that (1) implies (2). As the system (2.6) is partition regular, Lemma 2.1 implies that such a system must be infinitely partition regular, and the Inhomogeneous Rado's Theorem implies that there is a constant solution $s \in \mathbb{Z}$ to $A\vec{t} = \vec{d}$. Now let $c : \mathbb{N} \to [l]$ be any coloring and $r \in \mathbb{N}$; define $\chi : \mathbb{N} \to [l+1]$ as

$$\chi(x) = \begin{cases} c(x-s), & \text{if } x > s; \text{ or} \\ l+1, & \text{otherwise.} \end{cases}$$

Since the system (2.6) is infinitely partition regular, one can find χ-monochromatic $t_1, \ldots, t_n \in \mathbb{N}$ satisfying $A(t_1, \ldots, t_n) = d$ and $t_1, \ldots, t_n > s$ and, for each $i \in [k]$,

$$\vec{b_i} \cdot (t_1, \ldots, t_n) > r + \left| s \sum_{j=1}^{n} b_{ij} \right|.$$

Hence $t_1 - s, \ldots, t_n - s$ form a c-monochromatic solution to $A\vec{t} = 0$, and, for every $i \in [k]$, one has

$$\vec{b_i} \cdot (t_1 - s, \ldots, t_n - s) = \vec{b_i} \cdot (t_1, \ldots, t_n) - s \sum_{j=1}^{n} b_{ij} > r.$$

This shows that the system (2.7) is partition regular.

The proof that (2) implies (1) is similar: let $c : \mathbb{N} \to [l]$ be any coloring of $\mathbb{N}$ and let $r \in \mathbb{N}$. Define $\chi : \mathbb{N} \to [l+1]$ as

$$\chi(x) = \begin{cases} c(x+s), & \text{if } x > |s|; \text{ or} \\ l+1, & \text{otherwise.} \end{cases}$$

Let $t_1, \ldots, t_n > |s|$ be a χ-monochromatic solution of system (2.7) satisfying

$$\vec{b_i} \cdot \vec{t} > \left| \sum_{j=1}^{n} b_{i,j} s \right| + r$$

for all $i \in [k]$. Then $t_1 + s, \ldots, t_n + s$ are c-monochromatic, $A(t_1 + s, \ldots, t_n + s) = \vec{d}$ and, for all $i \in [n]$, one has

$$\vec{b_i} \cdot (t_1 + s, \ldots, t_n + s) = \vec{b_i} \cdot \vec{t} + \sum_{j=1}^{n} b_{i,j} s > r.$$

Hence the system (2.6) is partition regular.

Finally, the equivalence between (2) and (3) follows by Theorem 2.17 and Lemma 2.18. □

Following Observation 2.8, we write explicitly the system whose infinite partition regularity we need to check for later applications: Given an upper Rado functional $\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{m-1})$ for $P \in R[x_1, \dots, x_n]$, Theorems 2.7 and Theorem 2.19 prove that there exist $q_{m+1}, \dots, q_l \in \mathbb{Q}_{>0}$ such that the system

$$O(t_1, \dots, t_n, z_{m+1}, \dots, z_l) = \widehat{b}$$

is infinitely partition regular, where

$$O = \begin{pmatrix} \widehat{A}_{\mathcal{J}} & \mathbf{0}_{u\times v} \\ \begin{pmatrix} \alpha_{m,1} - \alpha_{m+1,1} \\ \vdots \\ \alpha_{m,1} - \alpha_{l,1} \end{pmatrix} & Q \end{pmatrix},$$

$\mathbf{0}_{u\times v}$ is the $u \times v$ ($u = k_0 + \dots + k_l - l + m - 1$ and $v = l - m$) matrix with 0 in all entries,

$$Q = \begin{pmatrix} -q_{m+1} & 0 & \dots & 0 \\ 0 & -q_{m+2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -q_l \end{pmatrix}$$

and $\widehat{b} \in \mathbb{Z}^{u+l-m-1}$ is defined as right after Definition 2.5.

Observe also that, when this happens with $m \geq 1$, the restrictions imposed by Corollaries 2.13 and 2.15 to $(J_0, \dots, J_l, d_0, \dots, d_{m-1})$ apply. We will take that into consideration to study polynomials in three variables in Section 4.

3 Rado conditions

In this section, we discuss some necessary and sufficient conditions for the partition regularity of equations that are formulated in terms of upper Rado functionals. Let us start by recalling the definition of the maximal Rado condition (see [4, Def'n. 2.16]).

Definition 3.1. Let $P(\boldsymbol{x}) = \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in \mathbb{Z}[x_1, \dots, x_n]$ and let

$$\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{m-1})$$

be an upper Rado functional for P. Setting $d_m = 0$, for all $q \in \mathbb{N}$ define the monovariate polynomial

$$Q_{\mathcal{J},q}(w) = \sum_{i=0}^{m} q^{d_i} \left(\sum_{\alpha \in J_i} c_\alpha w^{|\alpha|} \right).$$

We say that the polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]$ satisfies the *maximal Rado condition* if for all $q \in \mathbb{N} \setminus \{1\}$ there exists an upper Rado functional $\mathcal{J}$ for P such that $Q_{\mathcal{J},q}$ has a real root $1 \leq w \leq q$.

Let us note that when $m = 0$ (for example, when P is homogeneous) the Maximal Rado Condition gets the simpler form

$$\sum_{\alpha \in J_0} c_\alpha = 0.$$

When $m \geq 1$, by Corollary 2.15, there are $L_0, \ldots, L_m \in \mathbb{N}$ and $s \in \mathbb{Z} \setminus \{0\}$ such that for all $i \in \{0, \ldots, m\}$ and $\alpha \in J_i$, $L_i = |\alpha|$ and

$$Q_{\mathcal{J},q}(w) = w^{L_m} \sum_{i=0}^{m} q^{d_i} \left(\sum_{\alpha \in J_i} c_\alpha w^{\frac{d_i}{s}} \right).$$

Hence, P satisfies the maximal Rado condition if and only if for all $q \in \mathbb{N}$ there exist an upper Rado functional $\mathcal{F} = (J_0, \ldots, J_l, d_0, \ldots, d_{m-1})$ for P and $s \in \mathbb{Z}$ dividing $d_0, \ldots, d_{m-1}$ such that

$$R_{\mathcal{F},q}(w) = \sum_{i=0}^{m} q^{d_i} \bar{c}_i w^{\frac{d_i}{s}} = \sum_{i=0}^{m} \bar{c}_i \left(q w^{\frac{1}{s}} \right)^{d_i}$$

has a real root in $[1, q]$, where

$$\bar{c}_i = \sum_{\alpha \in J_i} c_\alpha.$$

The importance of the maximal Rado condition is that it gives a necessary condition for the partition regularity of polynomial equations over $\mathbb{N}$:

Theorem 3.2 (Theorem 3.1, [4]). *If $P \in \mathbb{Z}[x_1, \ldots, x_n]$ is a partition regular polynomial, then P satisfies the maximal Rado condition.*

However, in general, the maximal Rado condition alone is insufficient to prove the partition regularity of a given polynomial; actually, it is not even sufficient to prove that it has non-constant solutions.

Example 3.3. Let $P(x, y, z) = x^3 + y^3 - z^3$, $J_0 = \{(3, 0, 0), (0, 0, 3)\}$ and $J_1 = \{(0, 3, 0)\}$. Then, we have that

$$O = \begin{pmatrix} 3 & 0 & -3 & 0 \\ 3 & -3 & 0 & -1 \end{pmatrix}$$

satisfies the columns condition, which implies that the system

$$\begin{cases} 3t_1 & = & 3t_3 \\ 3t_1 - 3t_2 & = & \infty \end{cases}$$

is infinitely partition regular. Hence, $\mathcal{J} = (J_0, J_1)$ is an upper Rado functional for P of order 0. Moreover, for each $q \in \mathbb{N} \setminus \{1\}$ we have that

$$Q_{\mathcal{J},q}(w) = 1 - 1 = 0,$$

which proves that P satisfies the maximal Rado condition. However, $P(x, y, z) = 0$ is not partition regular, since Fermat's Last Theorem implies that this equation does not admit non-trivial natural solutions.

Therefore, a natural question that arises is: under which additional hypotheses is the maximal Rado condition sufficient to prove the partition regularity of a given equation? To answer this question, we introduce a strengthened notion.

Definition 3.4. A complete Rado functional is an upper Rado functional of the form $(J_0, \ldots, J_l, d_0, \ldots, d_{l-1})$, i.e. an upper Rado functional of maximal order.

Fixing $r \in R$, define $\exp_r : \mathbb{N} \to R$ as $\exp_r(x) = r^x$ and let $\overline{\exp}_r : \beta\mathbb{N} \to \beta R$ be the unique continuous extension of $\exp_r$ over $\beta\mathbb{N}$. Then, for $\mathcal{U} \in \beta\mathbb{N}$, we have that $U \in \overline{\exp}_r(\mathcal{U})$ if and only if $\{x \in \mathbb{N} : r^x \in U\} \in \mathcal{U}$. We say that a subset $S \subseteq R$ is closed under exponentiation if for all $r \in S$ one has $\exp_r[\mathbb{N}] \subseteq S$. It is easy to see that any set S that is closed under multiplication is also closed under exponentiation and that if S is closed under exponentiation, then $\overline{\exp}_s(\mathcal{U}) \in \beta S$ for all $s \in S$ and $\mathcal{U} \in \beta\mathbb{N}$.

Theorem 3.5. *Let $P(\boldsymbol{x}) = \sum_\alpha c_\alpha \boldsymbol{x}^\alpha \in R[x_1, \ldots, x_n]$, and let*

$$\mathcal{J} = (J_0, \ldots, J_l, d_0, \ldots, d_{l-1})$$

be a complete Rado functional for P. Define $d_l = 0$ and

$$Q_{\mathcal{J},P}(w) := \sum_{i=0}^{l} \overline{c_i} w^{d_i},$$

where for each $i \in [0, l]$ let $\overline{c}_i = \sum_{\alpha \in J_i} c_\alpha$. Suppose that $S \subseteq R$ is infinite and closed under exponentiation and that $Q_{\mathcal{J},P}$ has a root s in S. Then P is partition regular over S.

Proof. By Theorems 2.7 and 2.10, there exists a free $\mathcal{U} \in \beta\mathbb{N}$ that witnesses the infinite partition regularity of the system $\widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b}$, namely such that for all $U \in \mathcal{U}$, $i \in [0, l]$, $\alpha \in J_i$ and $\beta \in J_0$ there is $\vec{u} = (u_1, \ldots, u_n) \in U^n$ such that $(\alpha - \beta) \cdot \vec{u} = d_i$. Given any root $s \in S$ of $Q_{P,\mathcal{J}}$, we claim that $\mathcal{V} = \overline{\exp}_s(\mathcal{U})$ is a witness of the partition regularity of $P(x_1, \ldots, x_n) = 0$. Indeed, if $V \in \mathcal{V}$, we have that $U = \{x \in \mathbb{N} : s^x \in V\} \in \mathcal{U}$ and thus, as observed above, one can find $\vec{u} = (u_1, \ldots, u_n) \in U^n$ such that $(\alpha - \beta) \cdot \vec{u} = d_i$. For each $i \in [0, l]$, define $s_i = s^{u_i}$; then, $s_i \in V$ and

$$P(s_1, \ldots, s_n) = \sum_{i=0}^{l} \sum_{\alpha \in J_i} c_\alpha s_1^{\alpha(1)} \ldots s_n^{\alpha(n)} = \sum_{i=0}^{l} \sum_{\alpha \in J_i} c_\alpha s^{\alpha \cdot \vec{u}} = \sum_{i=0}^{l} \sum_{\alpha \in J_i} c_\alpha s^{\beta \cdot \vec{u} + d_i}$$

$$= s^{\beta\cdot\vec{u}}\sum_{i=0}^{l}\left(\sum_{\alpha\in J_i}c_\alpha\right)s^{d_i} = s^{\beta\cdot\vec{u}}\sum_{i=0}^{l}\overline{c}_i s^{d_i} = s^{\beta\cdot\vec{u}}Q_{\mathcal{J},P}(s) = 0,$$

as desired. □

Example 3.6. Let $a,\lambda\in\mathbb{N}$, $b\in\mathbb{C}$, and $P(x,y,z) = a^\lambda bxy^2 - (a^\lambda+b)x^2yz^\lambda + x^3z^{2\lambda}$. Let $J_2 = \{(1,2,0)\}$, $J_1 = \{(2,1,\lambda)\}$, $J_0 = \{(3,0,2\lambda)\}$, $d_1 = \lambda$ and $d_2 = 4\lambda$. Then $\mathcal{J} = (J_0, J_1, J_2, d_0, d_1)$ is an upper Rado functional for P of order 2, since the matrix

$$O = \begin{pmatrix} 1 & -1 & \lambda \\ 2 & -2 & 2\lambda \end{pmatrix}$$

is infinitely partition regular and the system $O\vec{t} = (2\lambda, 4\lambda)$ has a constant solution $s = 2$. We have that

$$Q_{\mathcal{J},P}(w) = a^\lambda b - (a^\lambda + b)w^{2\lambda} + w^{4\lambda}$$

has a as a natural root. Hence, we have that P is partition regular over $\mathbb{N}$ (following the proof of Theorem 2.7, we can derive that this equation is in fact partition regular over $\{a^n : n\in\mathbb{N}\}$. A similar result can also be proved for Example 3.7).

Example 3.7. For each $n,k\in\mathbb{N}$, let

$$P(x,y,z,u,w) = ax^n + by^n + cz^n - du^nw^{kn}.$$

For $a,b,c,d\in\mathbb{Z}$ such that $a+b\neq -c$. Let $J_0 = \{(0,0,0,n,kn)\}$ and

$$J_1 = \{(n,0,0,0,0),(0,n,0,0,0),(0,0,n,0,0)\}.$$

We claim that $\mathcal{J} = (J_0, J_1, rkn)$ is a complete Rado function for P, for any given $r\in\mathbb{N}$. Indeed, the matrix

$$A = \begin{pmatrix} -n & n & 0 & 0 & 0 \\ -n & 0 & n & 0 & 0 \\ -n & 0 & 0 & n & kn \end{pmatrix}$$

satisfies the columns condition and r is a constant solution of the system

$$A\cdot\begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ rkn \end{pmatrix}. \tag{3.1}$$

By Lemma 2.12, the system in (3.1) is infinitely partition regular over $\mathbb{N}$. Moreover,

$$Q_{\mathcal{J},P}(w) = (a+b+c)w^{rkn} - d.$$

Hence, if $s = d/(a+b+c)$ is an rkn-power in $\mathbb{N}$, then Theorem 3.5 states that $\exp_s(\mathcal{U})$ witnesses the PR of $P(x,y,z,u,w) = 0$ over $\mathbb{N}$ for any given choice of a free $\mathcal{U}\in\beta\mathbb{N}$ that witnesses the PR of the system (3.1).

The next result is an attempt to recover the partition regularity of a polynomial equation over an integral domain D from the existence of a root for $Q_{\mathcal{J},P}$ in the field of fractions of D. Ideally, this kind of result could be combined with tests for the existences of roots, such as the rational root test for unique factorization domains [1, Proposition 5.5], to provide the partition regularity of polynomial equations over these domains.

Lemma 3.8. *Let D be an integral domain and K be the field of fractions of D. Suppose that $\mathcal{J} = (J_0, \ldots, J_l, d_0, \ldots, d_{l-1})$ is a complete Rado functional for $P \in D[x_1, \ldots, x_n]$. In virtue of Corollaries 2.13 and 2.15, for each $i \in [0, l]$ let $L_i = |\alpha|$ for any $\alpha \in J_i$ and let $s \in \mathbb{Z}$ be such that whenever $i \geq 1$ one has*

$$L_i = L_0 + \frac{d_i}{s}.$$

Assume that $S \subseteq D$ is closed under multiplication and $a, b \in S$ are such that $b \neq 0$ and a/b is a root of $Q_{\mathcal{J},P}$. Define

$$\tilde{P}\left(x_1, \ldots, x_n\right) = P\left(\frac{x_1}{b^s}, \ldots, \frac{x_n}{b^s}\right) \in K\left[x_1, \ldots, x_n\right].$$

Then, $\tilde{P}$ is partition regular over S.

Proof. By Theorem 3.5, it suffices to produce a root for $Q_{\mathcal{J},\tilde{P}}$ in S. To this end, let us note that, if c'_α is the coefficient of $\tilde{P}$ associated to a exponent $\alpha \in J_i$, then $c'_\alpha = b^{-sL_i} c_\alpha$, where c_α is the coefficient of P associated with the multi-index $\alpha \in \operatorname{supp}(P)$. Hence,

$$Q_{\mathcal{J},\tilde{P}}(a) = \sum_{i=0}^{l} \bar{c}'_i a^{d_i} = \sum_{i=0}^{l} \bar{c}_i b^{-sL_i} a^{d_i}.$$

For each $i \in \{1, \ldots, l\}$ we have that $d_i = s(L_i - L_0)$, and thus

$$b^{sL_0} Q_{\mathcal{J},\tilde{P}}(a) = \sum_{i=0}^{l} \bar{c}_i b^{-s(L_i - L_0)} a^{d_i} = \sum_{i=0}^{l} \bar{c}_i \left(\frac{a}{b}\right)^{d_i} = Q_{\mathcal{J},P}\left(\frac{a}{b}\right) = 0,$$

which proves that a is a root for $Q_{\mathcal{J},\tilde{P}}$ in S, which concludes the proof. □

In the above lemma, we start from P and, under the given hypotheses, we deduce the partition regularity of $\widetilde{P}$. In some lucky cases, one could go back and use the partition regularity of $\widetilde{P}$ to deduce that of P. This is a general fact, for which we first need to set a definition.

Definition 3.9. Let S be a semigroup. An ultrafilter $\mathcal{U}$ over S is said to be S-divisible if for all $t \in S$, $tS \in \mathcal{U}$.

Let S be a semigroup and $\mathcal{U} \in \beta S$ be S-divisible. Then for each $s \in \mathbb{N}$ and $b \in S$ one has that $\mathcal{U} \in \overline{b^s S}$. Let $f : b^s S \to S$ be given by $f(b^s x) = x$; then, extending uniquely f to $\beta(b^s S) = \overline{b^s S}$, the ultrafilter $\mathcal{V} = f(\mathcal{U}) \in \beta S$ is well-defined and, for all $A \subseteq S$, it satisfies the relation $A \in \mathcal{V}$ if and only if $b^s A \in \mathcal{U}$. Given $s \in \mathbb{Z}$, let

$$\frac{\mathcal{U}}{b^s} = \begin{cases} b^s\mathcal{U}, & \text{if } s > 0 \\ \mathcal{U}, & \text{if } s = 0 \\ b^{-s}\mathcal{U}, & \text{if } s < 0. \end{cases}$$

Theorem 3.10. *With the same hypotheses and notation of Lemma 3.8, suppose that there exist $s \in \mathbb{Z}$, $b \in S$ and an S-divisible $\mathcal{U} \in \beta S$ that witnesses the partition regularity of $\tilde{P}(x_1, \dots, x_n) = 0$. Then P is partition regular over S.*

Proof. As $\mathcal{U}$ is S-divisible, we claim that the ultrafilter $\mathcal{V} = \frac{\mathcal{U}}{b^s}$ is a witness of the partition regularity of P; we divide the proof in the cases $s > 0$ and $s < 0$ (the case $s = 0$ is trivial).

Suppose that $s > 0$; as we observed above, for each $A \subseteq S$ one has that $A \in \mathcal{V}$ if and only if $b^s A \in \mathcal{U}$. Since $\mathcal{U}$ witnesses the partition regularity of $\tilde{P}$, one can find $b_1, \dots, b_n \in b^s A$ such that $\tilde{P}\left(b_1, \dots, b_n\right) = 0$; but each b_i can be rewritten as $b^s a_i$ for some $a_i \in A$, so, by the definition of $\tilde{P}$, it follows that $P\left(a_1, \dots, a_n\right) = 0$, hence $P\left(x_1, \dots, x_n\right) = 0$ has a solution in A.

If $s < 0$, let $k = -s$; note also that $\tilde{P}(x_1, \dots, x_n) = P(b^k x_1, \dots, b^k x_n)$. Now, one has that $A \in b^k\mathcal{U}$ if and only if

$$b^{-k}A = \{x \in S : b^k x \in A\} \in \mathcal{U}.$$

As $\mathcal{U}$ is a witness of the partition regularity of $\tilde{P}$, one can find $a_1, \dots, a_n \in b^{-k}A$ such that $\tilde{P}(a_1, \dots, a_n) = P(b^k a_1, \dots, b^k a_n) = 0$; since $b^k a_1, \dots, b^k a_n \in A$, we are done. □

Theorem 3.10 is a general fact that allows building new partition regular polynomials from known ones whose partition regularity is witnessed by S-divisible ultrafilters. This idea is not totally new: for example, when $S = \mathbb{N}$, $\mathbb{N}$-divisible ultrafilters are well-known objects related to divisibility relations between ultrafilters, see [11, 22] (in the context of the cited papers, such ultrafilters are called maximal, and several refined notions of maximality are studied), and the set of $\mathbb{N}$-divisible ultrafilters has very good algebraical properties; for example, it includes $\overline{K(\beta\mathbb{N}, \odot)}$ (see [11, Fact 1.7]). This is relevant, as all ultrafilters in $\overline{K(\beta\mathbb{N}, \odot)}$ witness the partition regularity of all partition regular homogeneous equations (and systems of equations, see e.g. [10, Theorem 2.4]). The following is a simple application of Theorem 3.10 on $\mathbb{N}$ based on the above observations.

Example 3.11. Let $\widetilde{P}(x, y, z, t) := xt_1 - y + zt_1t_2t_3$. This polynomial is partition regular and, by [10, Theorem 2.11], its partition regularity is witnessed by an ultrafilter in $\overline{K(\beta\mathbb{N}, \odot)}$, which is hence divisible. Then, by Theorem 3.10, for any $b \in \mathbb{N}_{\geq 1}$ one has that $b^2xt_1 - by + b^4zt_1t_2t_3$, is partition regular over $\mathbb{N}$.

4 Partition regularity of $P(x, y, z)$

Recently, in [2], we proved a necessary and sufficient condition for the partition regularity of systems of polynomial and functional equations in two variables. The methods introduced in [2] do not apply to equations in three variables, which are far from completely understood; indeed, only a few specific cases of families of equations are known to be (or not be) partition regular.

Our goal in this section is to use the machinery of Rado Conditions to find new conditions for the partition regularity problem for the three variables case.

4.1 Decomposition Theorems for Inhomogeneous P

By Corollary 2.15, the only possible upper Rado functionals for a homogeneous $P \in \mathbb{Z}[x_1, \ldots, x_n]$ are those of order $m = 0$; hence, the maximal Rado condition implies that if P is partition regular, then there is a non-empty subset of the coefficients of P that sums zero. As shown by the example below, a similar necessary condition for partition regularity still holds for equations of the form

$$P_1(x_1) + \cdots + P_n(x_n) = 0, \tag{4.1}$$

where $P_i \in \mathbb{Z}[x_i]$ for all $i \in [n]$ satisfies $P(0) = 0$.

Example 4.1 (Theorem 3.10, [10])**.** In [10] it was proven that, if $P_i(w) = \sum_{s=1}^{d_i} c_{i,s} w^s$ and the Equation (4.1) is partition regular over $\mathbb{N}$, then there is a non-empty $I \subseteq [n]$ such that

1. $d_{i_1} = d_{i_2}$ for all $i_1, i_2 \in I$, and
2. $\sum_{i \in I} c_{i,d_i} = 0$.

We can reprove the same result as a simple consequence of Theorem 3.2. Let us first observe that any upper Rado functional for P must have order $m = 0$; indeed, let $J \neq J'$ be Rado sets for P, $\alpha \in J$ and $\beta \in J'$, and suppose that the equation

$$(\alpha - \beta) \cdot (t_1, \ldots, t_n) = u$$

is infinitely partition regular over $\mathbb{N}$. Since the form of P forces each multi-index to have only one non-zero coordinate, this can only happen if $u = 0$ and $\alpha = \beta$, which is absurd. Then, Corollary 2.15 proves that the order of any upper Rado functional for P is 0. We now claim that, as a consequence, there is a non-empty $I \subseteq [n]$ satisfying

1. for any $i_1, i_2 \in I$ one has $d_{i_1} = d_{i_2}$; and
2. $\sum_{i \in I} c_{i,d_i} = 0$.

To prove this, let $\mathcal{J} = (J_0, \ldots, J_l)$ be an upper Rado function given by the maximal Rado condition (see Theorem 3.2). Since each multi-index of P has only one non-zero coordinate, Rado's Theorem implies that the Rado set J_0 is homogeneous: there exists $s \in \mathbb{N}$ such that for all $\alpha \in J_0$, the only non-zero coordinate of α is equal to s. We claim that, if $\alpha \in J_0$ appears in the polynomial P_i, then $s = d_i$. If not, let α_i be the multindex having the ith coordinate equals to d_i and all others equal to zero. By Theorem 2.7, one has that the system

$$\begin{cases} \widehat{A}_{\mathcal{J}}\vec{t} &= \widehat{b} \\ (\alpha - \alpha_i)\vec{t} &= \infty \end{cases}$$

is infinitely partition regular over $\mathbb{N}$. The inequality in the above system reads $(s - d_i)t_i = \infty$ so, by Theorem 2.19.(3), there is a positive $q \in \mathbb{Q}$ such that

$$(s - d_i)t_i - qz = 0$$

is partition regular (on the variables t_i and z) over $\mathbb{N}$; this is clearly false for the assumption $s - d_i < 0$. We conclude that

$$I = \{i \in [n] \mid \alpha_i \in J_0\}$$

is the desired set.

We observe that this condition is sufficient for the partition regularity in some cases (e.g., $x + y - z$), but it is not in others (e.g., $x^3 + y^3 - z^3$) and, sometimes, even unknown (e.g. $x^2 + y^2 - z^2$). We show below that conditions similar to those in Example 4.1 are necessary for all polynomial equations in three variables that only admit upper Rado functionals of order $m = 0$.

Following [10, Section 3], given a polynomial $P \in \mathbb{Z}[x_1, \dots, x_n]$ and multi-indexes $\alpha, \beta \in \operatorname{supp}(P)$, we write

- $\alpha \leq \beta$ if for all $i \in [n]$ one has $\alpha(i) \leq \beta(i)$;
- $\alpha < \beta$ if $\alpha \leq \beta$ and $\alpha \neq \beta$.

Finally, a set $J \subseteq \operatorname{supp}(P)$ is maximal if for all $\alpha \in J$ there is no $\beta \in \operatorname{supp}(P)$ satisfying $\alpha < \beta$.

Theorem 4.2. *Let $P \in \mathbb{Z}[x, y, z]$ be an inhomogeneous infinitely partition regular polynomial with no constant term that only admits upper Rado functionals of order $m = 0$, and so that in any such upper Rado functional $\mathcal{J} = (J_0, \dots, J_l)$ the set J_0 is homogeneous. Then, for each $i \in [0, l]$, the set J_i is homogeneous and there exist a homogeneous polynomials $H \in \mathbb{Z}[x, y, z]$ and a $Q \in \mathbb{Z}[x, y, z]$ such that*

1. *$P = H + Q$ and $\operatorname{supp}(P) \cap \operatorname{supp}(Q) = \emptyset$;*
2. *there is a non-empty $J \subseteq \operatorname{supp}(H)$ satisfying $\sum_{\alpha \in J} c_\alpha = 0$;*
3. *$\operatorname{supp}(H)$ is a maximal Rado set.*

To prove Theorem 4.2, we consider an arbitrary partition regular polynomial[4] $P \in \mathbb{Z}[x, y, z]$ and an arbitrary upper Rado functional

$$\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{m-1})$$

that satisfies the maximal Rado condition; we enumerate each J_i as $\{\alpha_{i,1}, \dots, \alpha_{i,k_i}\}$ and use Theorems 2.7 and 2.19 to prove the existence of positive $q_1, \dots, q_{l-m} \in \mathbb{Q}_{>0}$ such that the system

$$\begin{cases} \widehat{A}_{\mathcal{J}}\vec{t} & = & \widehat{b} \\ (\alpha_{m,1} - \alpha_{m+1,1}) \cdot \vec{t} - q_1 z_1 & = & 0 \\ \vdots & \vdots & \vdots \\ (\alpha_{m,1} - \alpha_{l,1}) \cdot \vec{t} - q_{l-m} z_{l-m} & = & 0 \end{cases}$$

is infinitely partition regular, where $\widehat{A}_{\mathcal{J}}$ was defined in the Equation (2.1) and $\widehat{b}$ in Page 6. In particular, by Lemma 2.12, this implies that the matrix O defined by[5]

$$\begin{pmatrix} \alpha_{0,2}(1) - \alpha_{0,1}(1) & \alpha_{0,2}(2) - \alpha_{0,1}(2) & \alpha_{0,2}(3) - \alpha_{0,1}(3) & 0 & 0 & \dots & 0 \\ \alpha_{0,3}(1) - \alpha_{0,1}(1) & \alpha_{0,3}(2) - \alpha_{0,1}(2) & \alpha_{0,3}(3) - \alpha_{0,1}(3) & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{0,k_0}(1) - \alpha_{0,1}(1) & \alpha_{0,k_0}(2) - \alpha_{0,1}(2) & \alpha_{0,k_0}(3) - \alpha_{0,1}(3) & 0 & 0 & \dots & 0 \\ \alpha_{1,2}(1) - \alpha_{1,1}(1) & \alpha_{1,2}(2) - \alpha_{1,1}(2) & \alpha_{1,2}(3) - \alpha_{1,1}(3) & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{l,k_l}(1) - \alpha_{l,1}(1) & \alpha_{l,k_l}(2) - \alpha_{l,1}(2) & \alpha_{l,k_l}(3) - \alpha_{l,1}(3) & 0 & 0 & \dots & 0 \\ \alpha_{0,1}(1) - \alpha_{m,1}(1) & \alpha_{0,1}(2) - \alpha_{m,1}(2) & \alpha_{0,1}(3) - \alpha_{m,1}(3) & 0 & 0 & \dots & 0 \\ \alpha_{1,1}(1) - \alpha_{m,1}(1) & \alpha_{1,1}(2) - \alpha_{m,1}(2) & \alpha_{1,1}(3) - \alpha_{m,1}(3) & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{m-1,1}(1) - \alpha_{m,1}(1) & \alpha_{m-1,1}(2) - \alpha_{m,1}(2) & \alpha_{m-1,1}(3) - \alpha_{m,1}(3) & 0 & 0 & \dots & 0 \\ \alpha_{m,1}(1) - \alpha_{m+1,1}(1) & \alpha_{m,1}(2) - \alpha_{m+1,1}(2) & \alpha_{m,1}(3) - \alpha_{m+1,1}(3) & -q_1 & 0 & \dots & 0 \\ \alpha_{m,1}(1) - \alpha_{m+2,1}(1) & \alpha_{m,1}(2) - \alpha_{m+2,1}(2) & \alpha_{m,1}(3) - \alpha_{m+2,1}(3) & 0 & -q_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_{m,1}(1) - \alpha_{l,1}(1) & \alpha_{m,1}(2) - \alpha_{l,1}(2) & \alpha_{m,1}(3) - \alpha_{l,1}(3) & 0 & 0 & \dots & -q_{l-m} \end{pmatrix}$$

must satisfy the columns condition and there is a constant solution $s \in \mathbb{Z}$ for $\widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b}$. Enumerating the columns of O as $C_1, \dots, C_{l-m+3}$, Rado's Theorem witnesses the existence of a partition $I_0, \dots, I_r$ of $[l - m + 3]$ such that

1. $\sum_{i \in I_0} C_i = \vec{0}$; and
2. for each $u \in [r]$, $\sum_{i \in I_u} C_i \in \operatorname{span}_{\mathbb{Q}}\{C_j : j \in I_0 \cup \dots \cup I_{u-1}\}$.

[4]Since in only one of the cases treated below we will observe that m can be different from 0, we decide to start with an arbitrary polynomial instead of assuming from the start that P only has upper Rado functionals of order $m = 0$. In each case below, this choice produces more information.

[5]The fact that, in the case where $m = 0$ or some J_i is a singleton, some lines of the matrix $\widehat{A}_{\mathcal{J}}$ will be empty does not affect the proofs.

By the configuration of the matrix O and without loss of generality, one of the following possibilities must happen:

Case 1. $I_0 \cap \{1, 2, 3\} = \{1\}$;
Case 2. $I_0 \cap \{1, 2, 3\} = \{1, 2\}$; or
Case 3. $I_0 \cap \{1, 2, 3\} = \{1, 2, 3\}$.

The rest of the proof exploits Rado's Theorem to give an exact expression to the exponents of the variables x, y, and z. Hence, we divide the proof of Theorem 4.2 in the lemmas below.

Lemma 4.3. *If Case 1 above happens, then the following facts hold:*

1. *$m = 0$ and for each $i \in [0, l]$ the set J_i is homogeneous and the exponent of x is constant in J_i; and*
2. *there exists a partition K_0, K_1 of $[0, l]$ such that*
   (a) *$J = \bigcup_{i \in K_0} J_i$ is an homogeneous set and the exponent of x is constant in J;*
   (b) *defining $H(\boldsymbol{x}) = \sum_{\alpha \in J} c_\alpha \boldsymbol{x}^\alpha$ and, for each $i \in K_1$, $Q_i(\boldsymbol{x}) = \sum_{\alpha \in J_i} c_\alpha \boldsymbol{x}^\alpha$, one has that $P = H + \sum_{i \in K_1} Q_i$ and $\operatorname{supp}(H)$ is maximal.*

Proof. Since $I_0 \cap \{1, 2, 3\} = \{1\}$, one has that for each $i \in [0, l]$ and $j \in [k_i]$

$$\alpha_{i,1}(1) = \alpha_{i,j}(1), \tag{4.2}$$

i.e. in each J_i the exponent of the variable x is constant, say equal to a_i. Since the first column of each $\widehat{A}_{\mathcal{J}}$ is zero, either the other two columns of $\widehat{A}_{\mathcal{J}}$ are zero, implying thus that P is constant (which, by our assumption, is absurd), or the sum of the columns 2 and 3 of $\widehat{A}_{\mathcal{J}}$ is zero. Hence, for each $i \in [l]$ and $j \in [k_i]$ one has that

$$\alpha_{i,1}(2) + \alpha_{i,1}(3) = \alpha_{i,j}(2) + \alpha_{i,j}(3), \tag{4.3}$$

i.e. the polynomial $Q_i \in \mathbb{Z}[y, z]$ given by

$$Q_i(y, z) = \sum_{\alpha \in J_i} c_\alpha y^{\alpha(2)} z^{\alpha(3)}$$

is homogeneous; additionally, by Equations (4.2) and (4.3) each set J_i is homogeneous. Moreover, since the first column of $\widehat{A}_{\mathcal{J}}$ is zero, if $i \leq m$, then we have that

$$\alpha_{i,1}(1) = \alpha_{m,1}(1),$$

so, if we call $a = \alpha_{m,1}(1)$, we can rewrite $P(x, y, z) = \sum_{i=0}^{l} x^a Q_i(y, z)$. Since the last two columns of $\widehat{A}_{\mathcal{J}}$ sum to zero,

$$\alpha_{i,1}(2) + \alpha_{i,1}(3) = \alpha_{m,1}(2) + \alpha_{m,1}(3),$$

i.e. the set $J_0 \cup \cdots \cup J_m$ is homogeneous; but, by Corollary 2.15, this can only happen if $m = 0$. We conclude that the upper Rado functional is of the form $(J_0, \ldots, J_l)$.

Let $u, v \in [r]$ be such that $2 \in I_u$ and $3 \in I_v$ and for each $i \in [l]$ let $u_i \in [0, r]$ be such that $i + 3 \in I_{u_i}$ (i.e. u_i is such that I_{u_i} is the cell containing the index of the column with all entries equal zero except for the entry containing $-q_i$). We divide the rest of the proof into the following cases: 1.a) $u \neq v$, or 1.b) $u = v$.

Case 1.a: $u \neq v$. In this case, Rado's Theorem implies that the second and the third columns of the matrix $\widehat{A}_{\mathcal{J}}$ are linear combinations of the first column, which is zero. By the format of the submatrix $\widehat{A}_{\mathcal{J}}$ of O, each Rado set J_i must be a singleton, say $J_i = \{\alpha_i\}$; since $m = 0$, the Maximal Rado Condition implies that $c_{\alpha_0} = 0$, which is absurd since $\alpha_0 \in \operatorname{supp}(P)$. Hence, this case cannot happen.

Case 1.b: $u = v$. In this case, we claim that if $u_i \leq u$, then either $u_i = 0$ or $u = u_i$. Indeed, if $0 < u_i < u$, one has that

$$\alpha_{0,1}(1) = \alpha_{i,1}(1).$$

By the format of the matrix O, one has that $q_i = 0$; this contradicts the fact that, by Theorem 2.19, $q_i > 0$. This finishes the proof of this case.

Define

$$K_0 = \{i \in [l] : u < u_i\} \cup \{0\} \quad \text{and} \quad K_1 = \{i \in [l] : u_i = 0 \text{ or } u_i = u\}.$$

Then the exponent of the variable x is constant, and equal to a, in the set $J = \cup_{i \in K_0} J_i$ and the polynomial

$$H(x, y, z) = x^a \sum_{i \in K_0} Q_i(y, z)$$

is homogeneous; moreover, since $m = 0$, the Maximal Rado Condition implies that $\overline{c}_0 = 0$; taking thus $J = J_0$, the fact that $0 \in K_1$ implies that $J \subseteq \operatorname{supp}(H)$ and $\sum_{\alpha \in F} c_\alpha = \overline{c}_0 = 0$.

Let $i \in K_1$. We prove that for any $\alpha \in \operatorname{supp}(H)$ and any $\beta \in J_i$, one cannot have $\alpha < \beta$. By the format of the matrix O, we need only to verify that $\alpha < \alpha_{i,1}$ cannot happen. If $u_i = 0$, then the index of the cell containing the column $-q_i$ is 0 and thus

$$\alpha_{0,1}(1) - \alpha_{i,1}(1) = q_i > 0.$$

Hence, $\alpha < \alpha_{i,1}$ cannot happen because the exponent of x is constant in H. If instead $u_i = u$, then this means that $\alpha_{0,1}(1) - \alpha_{i,1}(1) = 0$ and, consequently, one has that

$$[\alpha_{0,1}(2) - \alpha_{i,1}(2)] + [\alpha_{0,1}(3) - \alpha_{i,1}(3)] = q_i > 0,$$

which implies that either $\alpha_{0,1}(2) - \alpha_{i,1}(2) > 0$ or $\alpha_{0,1}(3) - \alpha_{i,1}(3) > 0$. Since H is homogeneous and the exponent of x is constant in H, for any $\alpha \in \operatorname{supp}(H)$ one has that $\alpha(2) - \alpha_{0,1}(2) = \alpha(3) - \alpha_{0,1}(3)$. Thus, either $\alpha(2) - \alpha_{i,1}(2) > 0$ or $\alpha(3) - \alpha_{i,1}(3) > 0$; for any other $j \in [k_i]$, one has that

$$\alpha_{i,j}(1) - \alpha_{i,1}(1) = \alpha_{i,j}(2) - \alpha_{i,1}(2).$$

We conclude that either $\alpha(2) - \alpha_{i,j}(2) > 0$ or $\alpha(3) - \alpha_{i,j}(3) > 0$ and thus $\operatorname{supp}(H)$ must be maximal. $\square$

Contrary to what happens in Cases 1 and 3, in Case 2, namely when $I_0 \cap \{1, 2, 3\} = \{1, 2\}$, we will have that either $m = 0$ or $m \geq 1$. For this case, when $m = 0$, we cannot guarantee that J_0 is homogeneous without any additional hypothesis (see Theorem 1). The next Lemma will be used in dealing with Case 2 to arrive at the conclusions of Theorem 4.2.

Lemma 4.4. *If Case 2 happens, we have that in each Rado set of $\mathcal{J}$, P is homogeneous in (x, y). Moreover, assuming that $m = 0$ and J_0 homogeneous, the conclusions of Theorem 4.2 holds in this case.*

Proof. Let us first fix $u \in [r]$ such that $3 \in I_u$. By Rado's Theorem, for each $a \in I := I_0 \cup \cdots \cup I_{u-1}$ there is a $\rho_a \in \mathbb{Q}$ such that $\sum_{i \in I_u} C_i = \sum_{a \in I} \rho_a C_a$.

Since $I_0 \cap \{1, 2, 3\} = \{1, 2\}$, given any $i \in [0, l]$ and any $j \in [k_i]$, analysing the matrix $M(J_i)$, we have that

$$\alpha_{i,j}(1) + \alpha_{i,j}(2) = \alpha_{i,1}(1) + \alpha_{i,1}(2) \tag{4.4}$$

and, thus

$$\begin{aligned} \alpha_{i,j}(3) - \alpha_{i,1}(3) = \rho_1[\alpha_{i,j}(1) - \alpha_{i,1}(1)] + \rho_2[\alpha_{i,j}(2) - \alpha_{i,1}(2)] = \\ = (\rho_1 - \rho_2)[\alpha_{i,j}(1) - \alpha_{i,1}(1)]. \end{aligned} \tag{4.5}$$

If $m \geq 1$, by Corollary 2.15 each J_i is homogeneous for $i \leq m$; if $m = 0$, we assumed J_0 to be homogeneous in the hypothesis of Theorem 4.2. Consequently, we have two cases, namely:

Case 2.a. $\rho_1 = \rho_2$; or

Case 2.b. for each $i \in [0, m]$ one has $\alpha_{i,1}(1) = \alpha_{i,j}(1)$.

Case 2.a. Let us first observe that

$$\alpha_{i,1}(1) + \alpha_{i,1}(2) = \alpha_{m,1}(1) + \alpha_{m,1}(2),$$

and

$$\alpha_{m,1}(3) - \alpha_{i,1}(3) = (\rho_1 - \rho_2)(\alpha_{m,1}(1) - \alpha_{i,1}(1)) = 0.$$

Thus the set $J_0 \cup \cdots \cup J_m$ is homogeneous; this fact combined with Corollary 2.16 implies that $m = 0$. Also, by Equations (4.4) and (4.5), each J_i is an homogeneous set and thus

$$Q_i(x, y, z) = \sum_{\alpha \in J_i} c_\alpha x^{\alpha(1)} y^{\alpha(2)} z^{\alpha(3)}$$

is homogeneous. Then one has $P = Q_0 + \sum_{i=1}^{l} Q_i$. The Maximal Rado Condition reduces to $\bar{c}_0 = 0$, thus the sum of the coefficients of Q_0 is zero. We prove that $\mathrm{supp}(Q_0)$ is maximal. Indeed, if not, there are $\alpha \in J_0$, $i \in [l]$ and a $\beta \in J_i$ such that $\alpha < \beta$. But this would make the equation

$$(\alpha - \beta) \cdot \vec{t} - qz = 0$$

not partition regular over $\mathbb{N}$ when $q \in \mathbb{Q}_{>0}$, as it would not even have solutions.

Case 2.b. In this case, we have that $\alpha_{i,1}(1) = \alpha_{i,j}(1)$ for every $i \in [0, m]$ and every $j \in [k_i]$. By Equations (4.4) and (4.5), we have that the Rado sets $J_1, \dots, J_m$ are singletons, namely $J_i = \{\alpha_i\}$. If $m = 0$ and J_0 is homogeneous, the Maximal Rado Condition implies that $c_{\alpha_0} = 0$, which is absurd since $\alpha_0 \in \operatorname{supp}(P)$. By the hypothesis of Theorem 4.2, if P has only upper Rado sets of order $m = 0$, this case cannot occur. □

Lemma 4.5. *If Case 3 happens, then $m = 0$ and there are $q_1, \dots, q_l \in \mathbb{Q}_{>0}$, homogeneous polynomials $R_0, \dots, R_l \in \mathbb{Z}[x, y, z]$ and a partition L_0, L_2 of $[0, l]$ such that $P(x, y, z) = \sum_{i=0}^{l} R_i(x, y, z)$, there is an nonempty subset of the coefficients of $\sum_{i \in L_0} R_i$ that sum to zero, and for each $i \in [l]$, $\deg R_i = \deg R_0 - \chi_i q_i$, where*

$$\chi_i = \begin{cases} 1, & \text{if } i \in L_0; \\ 0, & \text{if } i \in L_1. \end{cases}$$

Proof. We have that $I_0 \cap \{1, 2, 3\} = \{1, 2, 3\}$. Since P has no constant term, we have that $|\alpha|$ is constant through $J_0, \dots, J_m$; by Corollary 2.15, this can only happen if $m = 0$. Let $K_1 = \{i \in [0, l] : i + 3 \in I_0\}$ and $K_2 = [0, l] \setminus K_1$. Then, by the format of the matrix O, we have that for each $i \in [l]$ and $j \in [k_i]$

$$\alpha_{i,j}(1) + \alpha_{i,j}(2) + \alpha_{i,j}(3) = \alpha_{0,1}(1) + \alpha_{0,1}(2) + \alpha_{0,1}(3) - \chi_i q_i. \tag{4.6}$$

For each $i \in [0, l]$, define $R_i(\boldsymbol{x}) = \sum_{\alpha \in J_i} c_\alpha \boldsymbol{x}^\alpha$. Then, each Q_i is homogeneous and by Equation 4.6, $\deg R_i = \deg R_0 - \chi_i q_i$. Let

$$L_0 = \{i \in [l] : \chi_i = 0\} \cup \{0\} \text{ and } L_1 = [l] \setminus L_0.$$

Let $H = \sum_{i \in L_0} R_i$; by construction, H is homogeneous. The Maximal Rado Condition reduces to $\overline{c}_0 = \sum_{\alpha \in J_0} c_\alpha = 0$ and thus since $0 \in L_0$ that there is a non-empty $J = J_0$ is a non-empty subset of $\operatorname{supp}(H)$ whose sum of coefficients is zero. Moreover, since $\chi_i q_i > 0$ for all $i \in [l]$, one has that $\operatorname{supp}(H)$ is maximal. □

Example 4.6. It was shown in [9] that the equation $x + y = z^2$ is not partition regular; in fact, it is not even 3-partition regular[6] over $\mathbb{N}$ apart from the constant solution $x = y = z = 2$ [13, Theorem 1.1]. However, this equation is known to be 2-partition regular over $\mathbb{N}$ (see [13, Theorem 1.1] and [3, Theorem 1.1]); this fact also highlights the difference between k-partition regularity and partition regularity. The non-infinite partition regularity of this equation can also be seen as a trivial consequence of Theorem 4.2 (as well as of Example 4.1, of course). Indeed, the possible Rado sets of $P(x, y, z) = x + y - z^2$ are $J_x = \{(1, 0, 0)\}$, $J_y = \{(0, 1, 0)\}$, $J_z = \{(0, 0, 2)\}$ and $J_{xy} = \{(1, 0, 0), (0, 0, 1)\}$. Since $-t_1 + t_2 = u$ cannot be infinitely partition regular for any $u \in \mathbb{N}$ and the matrix $\begin{pmatrix} -1 & 0 & 2 \end{pmatrix}$ does not satisfy the columns condition, we must have that the only possible upper Rado functionals for

[6] Given an integer $k \geq 2$, we say that a system of equations is k-partition regular over $\mathbb{N}$ if any coloring c of $\mathbb{N}$ in k colors have monochromatic solutions.

P are those of order $m = 0$. Since P cannot be decomposed in $H + R$, where H is homogeneous with a subset of its coefficients summing zero, the given equation is not partition regular.

As shown in the proofs in this section, in Theorem 4.2 we assumed a technical hypothesis, namely the fact that in any upper Rado functional $\mathcal{J} = (J_0, \dots, J_l)$ the set J_0 is homogeneous, which holds automatically in Cases 1 and 3. We conjecture that the same actually happens also in Case 2 but, as we are not been able to prove it, we conclude this Section with the following open question:

Question 4.7. *Let $P \in \mathbb{Z}[x, y, z]$ be a partition regular inhomogeneous polynomial and suppose that any upper Rado function $\mathcal{J} = (J_0, \dots, J_l)$ given by the maximal Rado condition has order $m = 0$. Can J_0 be inhomogeneous?*

4.2 Complete Rado functionals in three variables

Having a complete Rado functional is neither necessary nor sufficient for the partition regularity, as the Schur equation $x + y = z$ is partition regular and does not admit a complete Rado functional and $xy^2 = 2z$ has a complete functional, namely $(\{(0, 0, 1)\}, \{(1, 2, 0)\}, 2)$, but it is not partition regular [10, Example 3.7]. Additionally, the Pythagorean equation $x^2 + y^2 = z^2$ does not admit a complete Rado functional and its partition regularity is unknown as of the time we write this paper.

The objective of this subsection is to classify all polynomials P in three variables over $\mathbb{Z}$ without constant term and having an inhomogeneous set of multi-indexes that admits a complete Rado functional, and to provide conditions under which such polynomials are partition regular.

Theorem 4.8. *An inhomogeneous $P \in \mathbb{Z}[x, y, z]$ admits a complete Rado functional $\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{l-1})$ with $l \geq 1$ if and only if, possibly after a permutation of the variables, there are naturals r and a, a rational ρ and a homogeneous $H \in \mathbb{Z}[u, v]$ such that $P(x, y, z) = z^{r-a\rho} H(xz^\rho, y)$.*

In this case, if $H(x, y) = \prod_{i=1}^{k} (a_i x - b_i y)$ is the decomposition of H into linear factors over $\mathbb{C}$, then P is partition regular over $\mathbb{N}$ if and only if there exists an $i \in [k]$ such that $\frac{a_i}{b_i}$ is a ρ-power in $\mathbb{Q}_{>0}$.

We prove the Theorem above as a consequence of the following results.

Lemma 4.9. *An inhomogeneous $P \in R[x, y, z]$ admits a complete Rado functional $\mathcal{J} = (J_0, \dots, J_l, d_0, \dots, d_{l-1})$ with $l \geq 1$ if and only if (possibly after a permutation of its variables) there is an enumeration $\alpha_0, \dots, \alpha_l$ of $\operatorname{supp}(P)$ such that*

1. *for each $i, j \in \{0, \dots, l\}$, $\alpha_i(1) + \alpha_i(2) = \alpha_j(1) + \alpha_j(2)$;*
2. *the sign of $\alpha_i(1) - \alpha_l(1)$ is constant; and*
3. *there exists $\rho \in \mathbb{Q}^\times$ such that, for each $i \in [0, l-1]$, $\alpha_i(3) = \alpha_l(3) + \rho[\alpha_i(1) - \alpha_l(1)]$ and $\rho[\alpha_i(1) - \alpha_l(1)] \neq 0$.*

Proof. Let $\mathcal{J} = (J_0, \ldots, J_l, d_0, \ldots, d_{l-1})$ be a complete Rado functional for P. For each $i \in [0, l]$, let $J_i = \{\alpha_{1,i}, \ldots, \alpha_{k_i,i}\}$; by Theorem 2.7 we have that $\widehat{A}_{\mathcal{J}}$ satisfies the columns condition and the system $\widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b}$ has a constant solution $s \in \mathbb{Z}$; let us also observe that $\widehat{A}_{\mathcal{J}}$ has three columns, namely C_1, C_2 and C_3. Since P is not homogeneous, there are two columns C_i and C_j of $\widehat{A}_{\mathcal{J}}$ that sum to zero and the third one, C_k, that is a $\mathbb{Q}$-linear combination of C_i and C_j; without loss of generality, we assume that $C_1 + C_2 = 0$ and there is a $\rho \in \mathbb{Q}$ ($\rho \neq 0$ since P is not homogeneous) such that $C_3 = \rho C_1$.

It is easy to see that the fact that $C_1 + C_2 = 0$ implies that $\alpha_{i,j}(1) + \alpha_{i,j}(2)$ is constant along the monomials of P; i.e. since P has no constant term, $P(x, y, 1)$ is a homogeneous polynomial of $R[x, y]$. By Corollary 2.14, one has that each J_i is a homogeneous Rado set. As a consequence, we have that $\alpha_{i,j}(3)$ is constant inside each J_i. We claim that J_i is a singleton; indeed, let $s \in \mathbb{Z}$ be a constant solution to the system $\widehat{A}_{\mathcal{J}}\vec{t} = \widehat{b}$. Then, for each $i \in [0, l-1]$ we have that

$$\begin{aligned} d_i &= s[\alpha_{i,1}(1) - \alpha_{l,1}(1)] + s[\alpha_{i,1}(2) - \alpha_{l,1}(2)] + s[\alpha_{i,1}(3) - \alpha_{i,1}(3)] \\ &= s\rho[\alpha_{i,1}(1) - \alpha_{l,1}(1)], \end{aligned} \tag{4.7}$$

thus $\alpha_{i,j}(1)$ is constant inside J_i; since $C_1 + C_2 = 0$, this also implies that $\alpha_{i,j}(2)$ is constant inside J_i, i.e. J_i is a singleton, say $J_i = \{\alpha_i\}$. Finally, by Corollary 2.15, we have that $|\alpha_i| = |\alpha_l| + \frac{d_i}{s}$ and thus by Equation (4.7),

$$\alpha_i(3) = \alpha_l(3) + \rho[\alpha_i(1) - \alpha_l(1)].$$

Since $d_i \in \mathbb{N}$, we must have $\rho[\alpha_i(1) - \alpha_l(1)] \neq 0$. The proof of the converse is direct and similar. □

Hence, if $\mathrm{supp}(P) = \{\alpha_0, \ldots, \alpha_l\}$ is an enumeration for $\mathrm{supp}(P)$ that satisfies conditions (1) and (2) above, there exist $c_0, \ldots, c_l \in \mathbb{Z}^\times$ such that

$$\begin{aligned} P(x, y, z) &= c_l x^{\alpha_l(1)} y^{\alpha_l(2)} z^{\alpha_l(3)} + \sum_{i=0}^{l-1} c_i x^{\alpha_i(1)} y^{\alpha_i(2)} z^{\alpha_i(3)} \\ &= c_l x^{\alpha_l(1)} y^{\alpha_l(2)} z^{\alpha_l(3)} + \sum_{i=0}^{l-1} c_i x^{\alpha_i(1)} y^{\alpha_i(2)} z^{\alpha_l(3)+\rho[\alpha_i(1)-\alpha_l(1)]} \\ &= c_l x^{\alpha_l(1)} y^{\alpha_l(2)} z^{\alpha_l(3)} + z^{\alpha_l(3)-\rho\alpha_l(1)} \sum_{i=0}^{l-1} c_i x^{\alpha_i(1)} y^{\alpha_i(2)} z^{\rho\alpha_i(1)} \\ &= z^{\alpha_l(3)-\rho\alpha_l(1)} \left(c_l x^{\alpha_l(1)} y^{\alpha_l(2)} z^{\rho\alpha_l(1)} + \sum_{i=0}^{l-1} c_i x^{\alpha_i(1)} y^{\alpha_i(2)} z^{\rho\alpha_i(1)} \right) \\ &= z^{\alpha_0(3)-\rho\alpha_0(1)} \sum_{i=0}^{l} c_i (xz^\rho)^{\alpha_i(1)} y^{\alpha_i(2)}. \end{aligned}$$

Hence, we have proven the first part of Theorem 4.8. We now proceed to give a condition under which such equations are partition regular. As a polynomial equation is partition regular if and only if one of the factors of the polynomial is (see e.g. [17, Theorem 3.7]), and since $z^{r-\rho a_0}$ is never 0 on $\mathbb{N}$, we are left with the problem of characterizing which equations of the form $H(xz^\rho, y) = 0$ are partition regular, where $H \in \mathbb{Z}[x, y]$ is homogeneous and $\rho \in \mathbb{Q}^\times$. Over $\mathbb{C}$, the homogeneity of H implies that H can be factorized as a product of linear factors, namely $H(x, y) = \prod_{i=1}^{k} (a_i x - b_i y)$; hence, the equation $H(xz^\rho, y) = 0$ is partition regular if and only if there is an $i \in [k]$ such that the equation $b_i y = a_i xz^\rho$ is partition regular over $\mathbb{N}$. Thus, it is enough to characterize the partition regularity of equations of the form

$$a^n x^n z^m = b^n y^n; \tag{4.8}$$

we settle this problem here in the case where $m, n, a, b \in \mathbb{Z}$ satisfy $\gcd(m, n) = \gcd(a, b) = 1$.

Lemma 4.10. *The Equation* (4.8) *is partition regular over* $\mathbb{N}$ *if and only if* $\frac{a}{b}$ *is an* $\frac{m}{n}$*-power in* $\mathbb{Q}_{>0}$.

Let us first note that if a^n and b^n have opposite signs, then Equation (4.8) is never partition regular over $\mathbb{N}$, so the request that $\frac{a}{b}$ is a $\mathbb{Q}_{>0}$ power cannot be relaxed to $\mathbb{Q}$.

Now, let us consider for each prime p the p-adic valuation $\nu_p : \mathbb{Q}^\times \to \mathbb{Z}$. Since ν_p is a completely additive function[7], for each non-zero rationals ρ and r such that $\rho^r \in \mathbb{Q}^\times$, we have that $\nu_p(\rho^r) = r\nu_p(\rho)$. As such, an irreducible fraction $\frac{a}{b}$ is an $\frac{m}{n}$-power, where $\frac{m}{n}$ is also irreducible, if and only if for all prime p one has that m divides $n\nu_p(\frac{a}{b})$.

Proof of Lemma 4.10. We proceed analogously to the proof of [12, Lemma 3.3]. Suppose that $r = \frac{a}{b}$ is not an $\frac{m}{n}$-power in $\mathbb{Q}$. Then, there is a prime p such that m does not divide $n\nu_p\left(\frac{a}{b}\right)$. For each $i \in [0, m-1]$, define

$$C_i = \{x \in \mathbb{Q} : n\nu_p(x) \equiv i \mod m\}.$$

If $x, y, z \in C_i$, then

$$\begin{aligned} n\nu_p(axz^{\frac{m}{n}}) - n\nu_p(by) &\equiv n\nu_p(r) + n\nu_p(x) + m\nu_p(z) - n\nu_p(y) \\ &\equiv n\nu_p(r) \not\equiv 0 \mod m \end{aligned}$$

which implies that the Equation (4.8) cannot be solved inside C_i. Hence, Equation (4.8) cannot be partition regular over $\mathbb{Q}$ (and, in particular, over $\mathbb{N}$).

Conversely, let us suppose that $\frac{b}{a} = \left(\frac{u}{v}\right)^{\frac{m}{n}}$, without loss off generality, for some $u, v, m, n \in \mathbb{N}$ such that $\gcd(u, v) = \gcd(m, n) = 1$. Given any coloring $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, we have a coloring $v\mathbb{N} = \bigcup_{i=1}^{r} v(C_i/u)$ of $v\mathbb{N}$. As $\{v^n \mid n \in \mathbb{N}\} \subseteq v\mathbb{N}$, we induce an r-coloring on $\mathbb{N}$ by letting $\Xi(n) = i$ if and only if $v^n \in \frac{v}{u}C_i$. By Rado's Theorem, we find $i \in [r]$ and distinct s_1, s_2, s_3 Ξ-monochromatic such that $ns_1 + ms_2 = ns_3$. By

[7] I.e. for each non-zero rationals a and b, $\nu_p(ab) = \nu_p(a) + \nu_p(b)$.

the definition of the coloring Ξ, one has that $x := uv^{s_1-1}, y := uv^{s_3-1}$ and $z := uv^{s_2-1}$ are elements of C_i; moreover,

$$x^n \cdot z^m = (uv^{s_1-1})^n \cdot (uv^{s_2-1})^m = \left(\frac{u}{v}\right)^n \left(\frac{u}{v}\right)^m v^{ns_1+ms_2} = \left(\frac{b}{a}\right)^n \left(\frac{u}{v}\right)^n v^{ns_3}$$
$$= \left(\frac{b}{a}\right)^n (uv^{s_3-1})^n = \left(\frac{b}{a}\right)^n y^n,$$

as desired. □

Example 4.11. By Lemma 4.10, the polynomial $P(x, y, z) = 4x^2z^2 - xy + 4xyz^2 - y^2 = (4xz^2 - y)(x + y)$ is partition regular over $\mathbb{N}$, while $Q(x, y, z) = 2x^2z^2 + 2xyz^2 - 3xy - 3y^2 = (2xz^2 - 3y)(x + y)$ is not partition regular over $\mathbb{N}$.

Example 4.12. The polynomial from Example 3.6 can be written as

$$P(x, y, z) = xH(xz^\lambda, y),$$

where $H(u, v) = u^2 - (a^\lambda + b)uv + a^\lambda bv^2 = (u - a^\lambda v)(u - bv)$. Since a^λ is a λ-power in $\mathbb{Q}_{>0}$, P is partition regular over $\mathbb{N}$.

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