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Hypercycle systems from semi-parallel classes

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Abstract

A 3-uniform 5-cycle $C(3,5)$, sometimes called a tight 5-cycle, consists of five vertices a,b,c,d,e and five 3-element sets abc, bcd, cde, dea, eab. A hypercycle system $\mathcal{C}(3,5,v)$ is a decomposition of the family of 3-element subsets of a v-element set in such a way that each part is isomorphic to $C(3,5)$ and each 3-set occurs in precisely one part. In this note we show a principle of recursion which can be used to build systems $\mathcal{C}(3,5,4v+1)$ and $\mathcal{C}(3,5,9v+1)$, and possibly more, when a certain kind of structural property is satisfied.

1 Introduction

In this paper we continue the study of edge decompositions of complete 3-uniform hypergraphs into 5-cycles.

An r-uniform hypercycle of length k ($k > r \geq 3$) is called a tight cycle if it is a cyclic sequence of k vertices of X in which any r consecutive vertices, and only those, form an edge. If r is understood, we simply call it a k-cycle. An r-uniform hypercycle of length k is denoted by $C(r,k)$.

A hypercycle system $\mathcal{C}(3,5,v)$ is a decomposition of the family of 3-element subsets of a v-element set in such a way that each part is isomorphic to $C(3,5)$ and each 3-set occurs in precisely one part. One very natural question is to determine the set of those v for which a $\mathcal{C}(3,5,v)$ exists.

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This problem was initiated by Meszka and Rosa [9] who noted that a necessary condition is $v \equiv 1, 2, 5, 7, 10, 11 \pmod{15}$. They also verified that there exists a 5-cycle system for every admissible $v \leq 17$ from these residue classes. This range was extended to $v \leq 22$ by Gionfriddo, Milazzo and Tuza [2], and to $v < 60$ by Keszler and Tuza [6], with some earlier intermediate constructions in a sequence of papers by Jirimutu et al.; see e.g. [8]. Beyond these cited works, which also contain infinite sequences of systems obtained by recursive constructions, a comprehensive account on the literature is given in [6]. For sufficiently large (huge) orders the existence of systems $\mathcal{C}(3, 5, v)$ follows from the very strong general results of Keevash's milestone paper [5] on Steiner systems.

In this paper our goal is to present some methods that are suitable for the construction of new infinite classes of 3-uniform 5-cycle systems. For this, in Sections 2, 3, and 4 we introduce three new types of systems.

For the first type—systems of order $4v + 1$ built upon four almost disjoint "semi-parallel classes" of systems of order $v + 1$—we create systems of all feasible orders under 60 (Section 2).

The second type is closely related to Steiner systems $\mathcal{S}(3, 5, v)$, they have the same arithmetic necessary conditions. The existence of $\mathcal{S}(3, 5, 41)$ is a famous old open problem in design theory, however here we are able to construct a "block-centered" cycle system for the order of 41 (Section 3).

The third type, "transversal cycle systems" are the cycle analogues of 3-wise transversal designs; the latter are widely used in Design Theory. Here we prove that transversal cycle systems can be constructed for every value of the group size parameter, demonstrating sharp contrast to transversal designs (Section 4).

Finally, in Section 5 we apply these structures to design a recursive construction of cycle systems. Namely, from special types of systems of orders $4v + 1$ and $w + 1$, a system of order $vw + 1$ is created. As a corollary, $\mathcal{C}(3, 5, v)$ systems are obtained for several new values of v.

1.1 Orbits of edges and triplet types

Symmetry is a useful tool in designing decompositions. In this paper we consider three types of symmetry:

- cyclic systems, where the vertex set is $\mathbb{Z}_v$ and the mapping defined for all $i \in \mathbb{Z}_v$ as $i \mapsto i + 1 \pmod{v}$ is an automorphism;
- 1-rotational systems, which are hypergraphs whose vertex set is $\mathbb{Z}_{v-1} \cup \{x\}$ and the mapping with fixed point x and $i \mapsto i + 1 \pmod{v - 1}$ for all $i \in \mathbb{Z}_{v-1}$ is an automorphism;
- hypergraphs whose vertex set is $\mathbb{Z}_{v-2} \cup \{x, y\}$ and the mapping with the two fixed points x and y, and taking $i \mapsto i + 1 \pmod{v - 2}$ for all $i \in \mathbb{Z}_{v-2}$ is an automorphism.

For these three kinds of symmetry we define triplet types over $\mathbb{Z}_m$, where $m = v$ or $m = v - 1$ or $m = v - 2$, respectively.

First, we need to define the *distance* of two vertices i, j as their shortest distance "along the cycle" in $\mathbb{Z}_m$, that is $||i-j|| = \min\{|i-j|, m-|i-j|\}$ for any two $i, j \in \mathbb{Z}_m$.

Consider any vertex triple $T = \{p, q, r\} \subset \mathbb{Z}_m$. Adopting a term from [2], the *difference triplets* associated with T are the triplets of nonnegative integers obtained in the following way. Find the increasing order of elements in T, i.e., let $0 \le a < b < c < m$ be such that $\{a, b, c\} = \{p, q, r\}$. Then, each of $(||a-b||, ||b-c||, ||c-a||)$, $(||b-c||, ||c-a||, ||a-b||)$, $(||c-a||, ||a-b||, ||b-c||)$ is considered as a difference triplet of T. These three are equivalent representations of T; usually (but not always), we take the lexicographically smallest of them. Viewing them as *cyclic* triplets, the three actually become identical.

A difference triplet of type (d, d, d') is also called a "symmetric difference"; and we use the term "reflected difference" for a pair $\{(d, d', d''), (d, d'', d')\}$ of difference triplets in which the three distances d, d', d'' are all distinct. In accordance with this, a *reflected cycle pair* means two cycles whose $5+5$ edges form five reflected differences.

Definition 1.1 Let $T = \{p, q, r\} \subset \mathbb{Z}_m$ be any edge in a 3-uniform hypergraph, where it is assumed that $i \mapsto i+1 \pmod m$ is an automorphism. Consider the lexicographically smallest difference triplet (d, d', d'') associated with T. (Here either of $d = d'$ and $d < d'$ may hold.) Then the *triplet type*, or simply the *type* of T is defined as the 3-tuple $(1, d+1, d+d'+1)$. $\diamond$

This notion, introduced in [6], is very useful in verifying constructions based on a concept of circular symmetry. Triplet types are invariant under the mapping $i \mapsto i+1 \pmod m$. Hence, the orbit of any T consists of all vertex triples having the same type.

2 Cycle systems with semi-parallel classes

Definition 2.1 Let $\mathcal{C}(3, 5, v)$ be a hypercycle system of order v with vertex set X, and let d and k be integers such that $v = kd+1$. A *semi-parallel d-class*, $\mathcal{SP}(d)$, for short, is a subsystem $\mathcal{C}' \subseteq \mathcal{C}(3, 5, v)$ satisfying the following conditions:

- $\mathcal{C}' = \mathcal{C}_1 \cup \cdots \cup \mathcal{C}_d$,
- each $\mathcal{C}_i$ is a $\mathcal{C}(3, 5, k+1)$ system,
- there exists a vertex $x \in \bigcap_{i=1}^{d} V(\mathcal{C}_i)$,
- the sets $V(\mathcal{C}_i) \setminus \{x\}$ for $i = 1, \ldots, d$ are mutually disjoint.

We call the sets $V(\mathcal{C}_i) \setminus \{x\}$ the *parts* of $\mathcal{C}'$, and vertex x the *center* of $\mathcal{C}'$. $\diamond$

It follows, in particular, that the subsystems $\mathcal{C}_i$ of $\mathcal{C}'$ cover the entire vertex set, but do not cover any vertex triple more than once. In the current context the relevant substructure will be $\mathcal{SP}(4)$, i.e., the case of $d = 4$.

Remark 2.2 If there exists a $\mathcal{C}(3,5,v)$ containing an $\mathcal{SP}(d)$, then inside the vertex set of every subsystem $\mathcal{C}_i$ one can take a system isomorphic to a fixed $\mathcal{C}(3,5,k+1)$. Hence it can be assumed without loss of generality that the vertices in all $V(\mathcal{C}_i)$ are labeled consistently and the label-preserving mapping $V(\mathcal{C}_i) \longleftrightarrow V(\mathcal{C}_j)$ (that maps x to itself) establishes an isomorphism between any two subsystems, $1 \le i,j \le d$. $\diamond$

Proposition 2.3 *If a 5-cycle system $\mathcal{C}(3,5,v)$ with an $\mathcal{SP}(4)$ subsystem exists, then*

$$v \equiv 1, 5, 17, 25, 37, 41 \pmod{60}.$$

Proof. We know that a $\mathcal{C}(3,5,v)$ system exists only if $v \equiv 1, 2, 5, 7, 10, 11 \pmod{15}$, which means

$$\begin{aligned} v \;\equiv\;& 1, 2, 5, 7, 10, 11, 16, 17, 20, 22, 25, 26, 31, \\ & 32, 35, 37, 40, 41, 46, 47, 50, 52, 55, 56 \qquad \pmod{60}. \end{aligned}$$

Moreover, the presence of $\mathcal{SP}(4)$ certainly requires that $v-1$ is a multiple of 4. This excludes all even numbers and all numbers of the form $4k+3$. Thus, the residue classes listed in the assertion remain. $\square$

More generally, one can obtain necessary conditions for the existence of $\mathcal{SP}(d)$ subsystems in $\mathcal{C}(3,5,kd+1)$ systems with other values of d, as well. However, we postpone those discussions to a later work, because for the 5-cycle systems constructed in Section 5 we apply $d = 4$ only.

In the rest of this section we show that all the four relevant feasible orders under 60 admit 5-cycle systems with $\mathcal{SP}(4)$ subsystems. We begin with the following observation concerning $\mathcal{S}(3,5,17)$.

Proposition 2.4 *The Steiner system $\mathcal{S}(3,5,17)$ contains four blocks which all contain a common point and are mutually disjoint otherwise.*

Proof. Let the set of points be $\mathbb{Z}_{17}$, and consider the cyclic 5-uniform hypergraph generated by the following four basic blocks:

$$\{0,1,7,10,16\}, \quad \{0,2,3,14,15\}, \quad \{0,4,6,11,13\}, \quad \{0,5,8,9,12\},$$

by taking all possible rotations modulo 17. Those $17 \times 4 = 68$ subsets of $\mathbb{Z}_{17}$ cover each 3-element set exactly once, and the four basic blocks above are mutually disjoint in $\mathbb{Z}_{17} \setminus \{0\}$. $\square$

This proposition implies that the smallest possible case of an $\mathcal{SP}(4)$ exists indeed.

Corollary 2.5 *The system $\mathcal{C}(3,5,17)$ derived from $\mathcal{S}(3,5,17)$ in* [2] *and* [8] *contains an $\mathcal{SP}(4)$ subsystem.*

There are several further values of v for which a cycle system $\mathcal{C}(3,5,v)$ with an $\mathcal{SP}(4)$ subsystem can be constructed, despite that arithmetic conditions exclude the existence of an $\mathcal{S}(3,5,v)$.

Cyclic representation. If an $\mathcal{SP}(4)$ is present in a cycle system $\mathcal{C}$, it is more convenient to assume that the order is $4v+1$, rather than v. Then the system is built upon four subsystems $\mathcal{C}_1,\dots,\mathcal{C}_4$; each $\mathcal{C}_\ell$ is a cycle system of order $v+1$ and has vertex set $V_\ell \cup \{x\}$, the sets $V_1,\dots,V_4$ being mutually disjoint. So $V(\mathcal{C}) = V_1 \cup V_2 \cup V_3 \cup V_4 \cup \{x\}$. We represent $V(\mathcal{C}) \setminus \{x\}$ as $\mathbb{Z}_{4v}$, where $V_\ell = \{i \in \mathbb{Z}_{4v} \mid i \equiv \ell \pmod{4v}\}$ for $\ell = 1,\dots,4$. Hence, the distances between elements other than x are computed in $\mathbb{Z}_{4v}$. Then the 3-element sets not covered by any cycles of $\mathcal{C}_1 \cup \dots \cup \mathcal{C}_4$ are:

- $\{x,a,b\}$ such that $a,b \in \mathbb{Z}_{4v}$ and $||a-b||$ is not divisible by 4;
- $\{a,b,c\}$ such that $a,b,c \in \mathbb{Z}_{4v}$ and at least one of $||a-b||$, $||b-c||$, $||c-a||$ is not divisible by 4.

The task is to decompose the family of these vertex triples into edge-disjoint 5-cycles. We carry out this with decompositions for which the mapping $i \mapsto i+1$ over $\mathbb{Z}_{4v}$, together with $x \mapsto x$, is an automorphism.

Next we present three such cycle systems.

Proposition 2.6 *There exist 5-cycle systems $\mathcal{C}(3,5,25)$, $\mathcal{C}(3,5,37)$, and $\mathcal{C}(3,5,41)$, that contain $\mathcal{SP}(4)$ subsystems.*

Proof. For all three cases we list the base cycles of the constructions. Following Definition 2.1 and the cyclic representation described above, x denotes the center of the $\mathcal{SP}(4)$ subsystem. In the three constructed systems $\mathcal{SP}(4)$ consists of four subsystems of order 7 or 10 or 11, respectively. We do not list their cycles here because they are well known systems from the cited references [2] and [9]. We only note that these subsystems cover

- for $v = 25$ the orbits of vertex triples $(x,0,4)$, $(x,0,8)$, $(x,0,12)$, $(0,4,8)$, $(0,4,12)$, $(0,4,16)$, $(0,8,16)$;
- for $v = 37$ the orbits of vertex triples $(x,0,4)$, $(x,0,8)$, $(x,0,12)$, $(x,0,16)$, $(0,4,8)$, $(0,4,12)$, $(0,4,16)$, $(0,4,20)$, $(0,4,24)$, $(0,4,28)$, $(0,8,16)$, $(0,8,20)$, $(0,8,24)$, $(0,12,24)$;
- for $v = 41$ the orbits of vertex triples $(x,0,4)$, $(x,0,8)$, $(x,0,12)$, $(x,0,16)$, $(x,0,20)$, $(0,4,8)$, $(0,4,12)$, $(0,4,16)$, $(0,4,20)$, $(0,4,24)$, $(0,4,28)$, $(0,4,32)$, $(0,8,16)$, $(0,8,20)$, $(0,8,24)$, $(0,8,28)$, $(0,12,24)$.

<u>$v = 25$.</u> Supplementing the four $\mathcal{C}(3,5,7)$ subsystems we take

(a) 3 base cycles containing x: $(x,0,1,13,2)$, $(x,0,3,15,6)$, $(x,0,5,17,10)$;

(b) 9 base cycles containing the symmetric triplets:
$(0, 1, 22, 2, 23)$, $(0, 2, 20, 4, 22)$, $(0, 3, 19, 5, 21)$,
$(0, 5, 14, 10, 19)$, $(0, 6, 23, 1, 18)$, $(0, 7, 10, 14, 17)$,
$(0, 9, 10, 14, 15)$, $(0, 10, 4, 20, 14)$, $(0, 11, 2, 22, 13)$;

(c) 3 reflected pairs of base cycles:
$(0, 11, 8, 17, 1)$, $(0, 2, 12, 23, 5)$, $(0, 16, 1, 18, 5)$ and
$(0, 8, 17, 14, 1)$, $(0, 6, 17, 3, 5)$, $(0, 11, 4, 13, 5)$.

$\underline{v = 37.}$ Supplementing the four $\mathcal{C}(3, 5, 10)$ subsystems we take

(a) 5 base cycles containing x: $(x, 0, 1, 19, 2)$, $(x, 0, 3, 21, 6)$, $(x, 0, 5, 23, 10)$, $(x, 0, 7, 25, 14)$, $(x, 0, 9, 27, 18)$;

(b) 12 base cycles containing the symmetric triplets:
$(0, 1, 34, 2, 35)$, $(0, 2, 32, 4, 34)$, $(0, 3, 30, 6, 33)$, $(0, 5, 26, 10, 31)$,
$(0, 6, 25, 11, 30)$, $(0, 7, 22, 14, 29)$, $(0, 10, 16, 20, 26)$, $(0, 11, 14, 22, 25)$,
$(0, 13, 10, 26, 23)$, $(0, 14, 8, 28, 22)$, $(0, 15, 6, 30, 21)$, $(0, 17, 2, 34, 19)$;

(c) 12 reflected pairs of base cycles:
$(0, 1, 5, 8, 10)$, $(0, 1, 6, 8, 17)$, $(0, 1, 7, 12, 15)$, $(0, 1, 8, 10, 21)$,
$(0, 1, 9, 14, 23)$, $(0, 1, 11, 35, 24)$, $(0, 2, 14, 19, 23)$, $(0, 3, 20, 32, 10)$,
$(0, 4, 11, 16, 27)$, $(0, 4, 22, 28, 15)$, $(0, 8, 25, 2, 18)$, $(0, 9, 19, 2, 16)$ and
$(0, 2, 5, 9, 10)$, $(0, 9, 11, 16, 17)$, $(0, 3, 8, 14, 15)$, $(0, 11, 13, 20, 21)$,
$(0, 9, 14, 22, 23)$, $(0, 25, 13, 23, 24)$, $(0, 4, 9, 21, 23)$, $(0, 14, 26, 7, 10)$,
$(0, 11, 16, 23, 27)$, $(0, 23, 29, 11, 15)$, $(0, 16, 29, 10, 18)$, $(0, 14, 33, 7, 16)$.

$\underline{v = 41.}$ Supplementing the four $\mathcal{C}(3, 5, 11)$ subsystems we take

(a) 5 base cycles containing x: $(x, 0, 1, 21, 2)$, $(x, 0, 3, 23, 6)$, $(x, 0, 5, 25, 10)$, $(x, 0, 7, 27, 14)$, $(x, 0, 9, 29, 18)$;

(b) 15 base cycles containing the symmetric triplets:
$(0, 1, 38, 2, 39)$, $(0, 2, 36, 4, 38)$, $(0, 3, 34, 6, 37)$, $(0, 5, 32, 8, 35)$,
$(0, 6, 28, 12, 34)$, $(0, 7, 26, 14, 33)$, $(0, 9, 22, 18, 31)$, $(0, 10, 21, 19, 30)$,
$(0, 11, 19, 21, 29)$, $(0, 13, 14, 26, 27)$, $(0, 14, 12, 28, 26)$, $(0, 15, 8, 32, 25)$,
$(0, 17, 6, 34, 23)$, $(0, 18, 4, 36, 22)$, $(0, 19, 2, 38, 21)$;

(c) 15 reflected pairs of base cycles:
$(0, 1, 5, 8, 18)$, $(0, 1, 6, 3, 11)$, $(0, 1, 7, 11, 16)$, $(0, 1, 8, 3, 15)$,
$(0, 1, 9, 15, 22)$, $(0, 1, 10, 15, 29)$, $(0, 2, 9, 12, 27)$, $(0, 2, 17, 6, 20)$,
$(0, 2, 18, 31, 12)$, $(0, 3, 21, 10, 17)$, $(0, 4, 14, 32, 19)$, $(0, 5, 22, 32, 17)$,
$(0, 6, 15, 38, 11)$, $(0, 6, 16, 37, 21)$, $(0, 9, 25, 39, 16)$ and
$(0, 10, 13, 17, 18)$, $(0, 8, 5, 10, 11)$, $(0, 5, 9, 15, 16)$, $(0, 12, 7, 14, 15)$,
$(0, 7, 13, 21, 22)$, $(0, 14, 19, 28, 29)$, $(0, 15, 18, 25, 27)$, $(0, 14, 3, 18, 20)$,
$(0, 21, 34, 10, 12)$, $(0, 7, 36, 14, 17)$, $(0, 27, 5, 15, 19)$, $(0, 25, 35, 12, 17)$,
$(0, 13, 36, 5, 11)$, $(0, 24, 5, 15, 21)$, $(0, 17, 31, 7, 16)$.

To facilitate the verification of correctness that indeed the required cycle systems are obtained, we provide more details on the base cycles in Section 6. □

3 Block-centered cycle systems

The following type is an intermediate structure between cycle systems and Steiner systems.

Definition 3.1 We say that a hypercycle system $\mathcal{C} = \mathcal{C}(3,5,v)$ with vertex set X is *block-centered* if there is an $x \in X$ such that the cycles incident with x are in complementary pairs; i.e., if (x,p,q,r,s) is a 5-cycle in $\mathcal{C}$, then also (x,q,s,p,r) is a 5-cycle in $\mathcal{C}$. Every such unordered 5-tuple $\{x,p,q,r,s\}$ will be called a *block* of $\mathcal{C}$. The collection of blocks will be denoted by $\mathcal{B}[\mathcal{C}]$. ◇

Since every vertex triple occurs in precisely one 5-cycle, it follows by definition that the derived system

$$\{B \setminus \{x\} \mid B \in \mathcal{B}[\mathcal{C}]\}$$

is a Steiner system $\mathcal{S}(2,4,v-1)$, whenever $\mathcal{C}$ is a block-centered 5-cycle system.

Proposition 3.2 *If a block-centered* $\mathcal{C}(3,5,v)$ *exists, then*

$$v \equiv 2, 5, 17, 26, 41, 50 \pmod{60}.$$

Proof. Recall that if $\mathcal{C}$ is a block-centered $\mathcal{C}(3,5,v)$ system, then $\mathcal{B}[\mathcal{C}]$ is an $\mathcal{S}(2,4,v-1)$ system. A classical result of Hanani [3] states that the latter exists if and only if $v \equiv 2,5 \pmod{12}$. Modulo 60 this means $v \equiv 2,14,26,38,50 \pmod{60}$ or $v \equiv 5,17,29,41,53 \pmod{60}$. On the other hand, the analogous necessary condition for $\mathcal{C}(3,5,v)$ systems is $v \equiv 1,2,5,7,10,11 \pmod{15}$, as stated in [9, 2]. Hence from 2, 14, 26, 38, 50 there remain 2, 26, 50 and from 5, 17, 29, 41, 53 there remain 5, 17, 41. □

Observe that the above six residue classes are exactly the ones derived from the divisibility conditions for $\mathcal{S}(3,5,v)$ systems. Hence, as a particular case of Keevash's theorem [5], for all sufficiently large v a block-centered 5-cycle system exists if and only if the divisibility conditions are satisfied. To put it in another way, for large v the existence of block-centered 5-cycle systems is equivalent to the existence of Steiner systems $\mathcal{S}(3,5,v)$. On the other hand, however, for many small values of v the existence of $\mathcal{S}(3,5,v)$ systems is an unsolved problem, and it may happen that either the spectrum of the two kinds of structures is not the same, or a block-centered 5-cycle system is easier to construct than a Steiner system $\mathcal{S}(3,5,v)$. In this way, based on a table of [1] for $v < 200$ the following problem arises naturally.

Problem 3.3 *Does there exist a block-centered* 5*-cycle system of order* v *for*

$$v = 41, 50, 62, 77, 86, 110, 122, 125, 137, 146, 161, 170, 182, 185, 197\,?$$

Solving this problem in full generality seems to be very hard. On the other hand we next show that the first case, $v = 41$, admits an affirmative answer.

Proposition 3.4 *There exists a block-centered* $\mathcal{C}(3,5,41)$ *system.*

Proof. We construct a $\mathcal{C}(3,5,41)$ that contains two central vertices x, y (rather than just one), and whose other vertices are labeled with the elements of $\mathbb{Z}_{39}$. The mapping that fixes x and y and acts as $i \mapsto i+1$ for all $i \in \mathbb{Z}_{39}$ will be an automorphism. We denote this special mapping as ϕ^*. The basis of the system consists of blocks of size 5 (i.e., 5-element subsets of $\{x,y\} \cup \mathbb{Z}_{39}$) and 5-cycles, (copies of $C(3,5)$) as follows:

(a) the block $(x, y, 0, 13, 26)$;

(b) the $3+3$ blocks $(x, 0, 1, 6, 31)$, $(x, 0, 2, 12, 23)$, $(x, 0, 3, 7, 22)$ and $(y, 0, 1, 9, 34)$, $(y, 0, 2, 18, 29)$, $(y, 0, 3, 20, 35)$;

(c) the 9 blocks in $\mathbb{Z}_{39}$: $(0, 1, 35, 4, 38)$, $(0, 2, 31, 8, 37)$, $(0, 3, 29, 10, 36)$, $(0, 5, 21, 18, 34)$, $(0, 6, 20, 19, 33)$, $(0, 7, 27, 12, 32)$, $(0, 9, 22, 17, 30)$, $(0, 11, 24, 15, 28)$, $(0, 14, 23, 16, 25)$;

(d) the cycles (12 reflected pairs) in $\mathbb{Z}_{39}$:
$(0, 1, 3, 7, 15)$, $(0, 1, 7, 10, 18)$,$(0, 1, 8, 3, 12)$, $(0, 1, 10, 17, 21)$, $(0, 1, 13, 31, 11)$, $(0, 2, 14, 37, 13)$,$(0, 3, 25, 32, 14)$,$(0, 4, 18, 8, 23)$, $(0, 5, 17, 34, 11)$,$(0, 6, 21, 3, 15)$,$(0, 8, 22, 37, 20)$,$(0, 9, 20, 3, 19)$ and $(0, 8, 12, 14, 15)$,$(0, 8, 11, 17, 18)$,$(0, 9, 4, 11, 12)$,$(0, 4, 11, 20, 21)$, $(0, 19, 37, 10, 11)$,$(0, 15, 38, 11, 13)$,$(0, 21, 28, 11, 14)$,$(0, 15, 5, 19, 23)$, $(0, 16, 33, 6, 11)$,$(0, 12, 33, 9, 15)$,$(0, 22, 37, 12, 20)$,$(0, 16, 38, 10, 19)$.

The orbits of (a) under ϕ^* cover all of the following vertex triples: those containing x and y together; one of x and y together with each pair of elements $i, i+13 \in \mathbb{Z}_{39}$; and each $\{i, i+13, i+26\} \subset \mathbb{Z}_{39}$.

Omitting x from the three blocks of (b) containing x we obtain 4-tuples that cover each of the 18 distances $\{1, 2, \ldots, 19\} \setminus \{13\}$ exactly once, thus the orbits of these blocks together with (a) cover all vertex triples (x, i, j) where $i, j \in \mathbb{Z}_{39}$. The analogous property holds for y as well. In addition, the vertex triples covered inside $\mathbb{Z}_{39}$ by the blocks of y are the reflected images of those covered by the blocks of x.

The nine blocks of (c) are symmetric and cover every vertex triple in which two distances between the three elements are equal.

After (a), (b), (c) the remaining uncovered vertex triples are all inside $\mathbb{Z}_{39}$, and form reflected pairs. They are covered with the orbits of the base cycles listed in (d). In order to make checking easier, we provide a detailed table in Section 6. □

The following problem is also natural to raise, although currently its solution seems to be far out of reach. In the light of Proposition 3.4, even the case of $v = 41$ remains a famous unsolved problem in the theory of Steiner systems.

Problem 3.5 *Determine the set of those orders v that admit a block-centered 5-cycle system $\mathcal{C}(3,5,v)$, but for which a $\mathcal{S}(3,5,v)$ system does not exist.*

4 Transversal cycle systems and 3-wise transversal designs

First, let us quote the definition of a well-known structure from the literature of design theory. It is a special type of 5-partite hypergraph; according to standard terminology, the partite classes are called groups.

Definition 4.1 A 3-*wise transversal design* of *block size* 5 and *group size* w is a 3-tuple $(X, G, \mathcal{B})$ where X is the vertex set of cardinality $5w$, G is a partition of X into five *groups* (classes) of size w each, and $\mathcal{B}$ is a family of w^3 *blocks*, which are 5-element subsets of X intersecting each group of G, with the property that every 3-tuple with elements from three distinct groups is contained in exactly one block. The common notation is 3-TD$(5, w)$; we simply write 3-TD when block size 5 and group size w are understood. $\diamond$

Denoting the groups as $X_i = \{x_{i,0}, x_{i,1}, \ldots, x_{i,w-1}\}$ for $1 \le i \le 5$, in the paper [2] the 3-TD with blocks

$$(x_{1,p}, x_{2,q}, x_{3,r}, x_{4,p+q+r}, x_{5,p+2q+3r})$$

is applied, where p, q, r are any three elements of $\mathbb{Z}_w$ and subscript addition is taken modulo w. This system exists whenever $w \equiv 1, 5 \pmod 6$. Concerning the method presented below, the applicable residue classes are

$$w \equiv 1, 19, 25, 31, 49, 55 \pmod{60}$$

because they may admit cycle systems $\mathcal{C}(3, 5, w + 1)$.

Further sufficient conditions for the existence of 3-TD$(5, w)$ systems are given by Hanani [4]. It follows, in particular, that a 3-TD$(5, w)$ exists also for $w = 4$ and $w = 9$. More generally, any product w of prime powers $q_i \equiv 0, 1, 4, 6, 9, 10$ (mod 15) may be of interest in this context. Note that the set $\{0, 1, 4, 6, 9, 10\}$ of residue classes is closed under multiplication modulo 15.

On the other hand, from the negative answer to *Euler's 36 Officers Problem* it follows that a 3-TD$(5, 6)$ does not exist (see, e.g., page 12 of [1]). Nevertheless, we can offer a method that is applicable for $\mathcal{C}(3, 5, v)$ systems also in those cases where 3-wise transversal designs cannot be constructed. For this purpose we introduce the following definition, that we formulate with general parameters, although here we will apply it only for $C(3, 5)$.

Definition 4.2 An r-*uniform transversal* k-*cycle system with group size* w, denoted as r-TC(k, w), is a 3-tuple $(X, G, \mathcal{C})$ where X is the vertex set of cardinality kw, G is a partition of X into k *groups* (classes) of size w each, and $\mathcal{C}$ is a family of r-uniform k-cycles, such that each cycle $C \in \mathcal{C}$ meets each group (in exactly one vertex) and each r-element subset of X meeting exactly r groups is contained in precisely one cycle.

More generally, an r-*uniform group-divisible* k-*cycle system with* g *groups and group size* w, denoted as r-GDC(k, g, w), is a 3-tuple $(X, G, \mathcal{C})$ where X is the vertex set of cardinality gw, G is a partition of X into g *groups* (classes) of size w each, and $\mathcal{C}$ is a family of r-uniform k-cycles, such that each cycle $C \in \mathcal{C}$ meets each group in at most one vertex, and each r-element subset of X meeting exactly r groups is contained in precisely one cycle. $\diamond$

It follows that the number of cycles is $\binom{k}{r}w^r/k$ in r-TC(k,w) and $\binom{g}{r}w^r/k$ in r-GDC(k,g,w). One should emphasize that the vertices in a cycle appear in a prescribed order, whereas the blocks of transversal designs are unordered sets.

The existence of r-TC and r-GDC systems with general parameters will be studied in the forthcoming paper [7]. Here we prove a construction for the case that is relevant in the present discussion; i.e., $g = k = 5$ and $r = 3$. It demonstrates that transversal cycle designs offer substantially more flexibility than 3-wise transversal designs do.

Theorem 4.3 (Transversal 5-Cycles Lemma.) *A* 3-TC$(5,w)$ *system with* $2w^3$ *cycles of length* 5 *exists for every* $w \geq 1$.

Proof. Let $X_1 \cup \cdots \cup X_5 = X$ be the partition of the vertex set into five groups. We are going to construct two collections of 5-cycles. The generic form of cycles of the first type is $C = x_1x_2x_3x_4x_5$, where $x_i \in X_i$ holds for all $1 \leq i \leq 5$. The generic form for the second type is $C = x_1x_3x_5x_2x_4$. Clearly, if the types of two cycles are not the same, then they are edge-disjoint. Hence it suffices to construct a system of the first type and apply an isomorphism to derive the system of the second type.

Let W be a quasigroup of order w. We label the vertices of each X_i with the elements of W. Now, for each $(a,b,c) \in W^3$, construct the 5-cycle

$$(x_1, x_2, x_3, x_4, x_5) = (a, b, c, a+b, b+c) \, .$$

It has to be verified that cyclically any three consecutive elements $x_i, x_{i+1}.x_{i+2}$ (where $x_6 := x_1$ and $x_7 := x_2$) uniquely determine the triplet (a,b,c). Indeed,

- $i = 1 \longrightarrow$ all the three of a, b, c are specified;
- $i = 2 \longrightarrow$ b and c are specified, and $a = x_4 - b$;
- $i = 3 \longrightarrow$ c is specified, $b = x_5 - c$, and $a = x_4 - b$;
- $i = 4 \longrightarrow$ a is specified, $b = x_4 - a$, and $c = x_5 - b$;
- $i = 5 \longrightarrow$ a and b are specified, and $c = x_5 - b$.

Thus, every 3-element set meeting three consecutive groups (vertex classes) is an edge in a cycle of the above collection. No vertex triple can be covered more than once, because the w^3 cycles contain exactly $5w^3$ edges.

The cycles of the other type are obtained in the same way, by the analogous rule $(x_1, x_3, x_5, x_2, x_4) = (a, b, c, a+b, b+c)$. $\square$

5 Recursive construction

The papers cited in the Introduction present some ways of building cycle systems from smaller ones. The following construction provides a further useful tool.

Theorem 5.1 (Recursion $v+1, w+1 \longrightarrow vw+1$.) *Suppose that there exists each of the following structures:*

- *a cycle system $\mathcal{C}(3,5,4v+1)$ containing a $\mathcal{SP}(4)$ subsystem;*
- *a block-centered cycle system $\mathcal{C}(3,5,w+1)$.*

Then there also exists a $\mathcal{C}(3,5,vw+1)$.

Proof. We combine a recursive construction of Hanani [4] for Steiner systems with ideas from the proof of Theorem 4.3 in [2]. Let $\mathcal{C}_{4v+1}$ be a $\mathcal{C}(3,5,4v+1)$ system in which a $\mathcal{SP}(4)$ subsystem $\mathcal{C}'$ with center x and parts X_1, X_2, X_3, X_4 is fixed. Hence each $X_i \cup \{x\}$ induces a $\mathcal{C}(3,5,v+1)$.

Consider now a block-centered $\mathcal{C}(3,5,w+1)$ system $\mathcal{C}_{w+1}$ whose vertex set is $\{0,1,\dots,w\}$. Assume that the blocks of $\mathcal{B}[\mathcal{C}_{w+1}]$ are incident with 0. We construct a system $\mathcal{C}^* = \mathcal{C}(3,5,vw+1)$ on the vertex set

$$V_{vw+1} := \{\infty\} \cup \{(i,j) \mid 1 \le i \le w,\ 1 \le j \le v\}$$

in the following way. Let C be any 5-cycle of $\mathcal{C}_{w+1}$.

- If $0 \notin V(C)$, say $C = (i_1, i_2, i_3, i_4, i_5)$ with all of its vertices being nonzero, we take a transversal cycle system 3-TC$(5,v)$, denoted by $\mathcal{T}_C$, with group size v and block size 5, whose five partition classes are

  $$Y_{i_k} := \{(i_k, j) \mid 1 \le j \le v\}, \qquad 1 \le k \le 5.$$

  Lemma 4.3 guarantees that this 3-TC$(5,v)$ exists. Then, for each cycle $C' \in \mathcal{T}_C$ we specify the 3-uniform 5-cycle

  $$(C' \cap Y_{i_1}, C' \cap Y_{i_2}, C' \cap Y_{i_3}, C' \cap Y_{i_4}, C' \cap Y_{i_5}),$$

  i.e., its edges are the vertex triples $(C' \cap Y_{i_k}, C' \cap Y_{i_{k+1}}, C' \cap Y_{i_{k+2}})$ for $k = 1,\dots,5$; addition in the sub-subscript is taken modulo 5.

  Note that this step does not need the cyclic sequence $(i_1, i_3, i_5, i_2, i_4)$ to be a cycle of $\mathcal{C}_{w+1}$, only the properties of a transversal cycle design were needed.

- If $0 \in V(C)$, say $C = (0, i_1, i_2, i_3, i_4)$, we create a 5-cycle system $\mathcal{C}[C]$ on the vertex set

  $$Z[C] := \{\infty\} \cup \{(i_k, j) \mid 1 \le k \le 4,\ 1 \le j \le v\}$$

  by taking a bijective mapping $\varphi_C : V(\mathcal{C}_{4v+1}) \to Z[C]$ such that $\varphi_C(x) = \infty$, moreover $\varphi_C(x_{i,j}) = (i_k, j)$ for all $1 \le i, k \le 4$ and all $1 \le j \le v$. Further, we also require that if $V(C') = V(C'')$ then $\varphi_{C'} = \varphi_{C''}$.

  Note that these mappings are consistent for all C, because the derived system $\{B \setminus \{0\} \mid 0 \in B \in \mathcal{B}[\mathcal{C}_{w+1}]\}$ is a Steiner system $\mathcal{S}(2,4,w)$.

We have to show that each vertex triple T in V_{vw+1} occurs in precisely one of the 5-cycles defined above. For $i = 1, \dots, w$ let us call the 5-element set $\{\infty, (i,1), (i,2), \dots, (i,v)\}$ as *fiber* i. Note that all vertex triples originating from cycles $C \in \mathcal{C}_{w+1}$ with $0 \notin C$ meet exactly three fibers.

- If T is contained in a fiber, then it occurs in a 5-cycle originating from $\mathcal{C}'$, the fixed $\mathcal{SP}(4)$ of $\mathcal{C}_{4v+1}$; there is precisely one 5-cycle for T. Indeed, due to the condition $\varphi_C(x_{i,j}) = (i_k, j)$, every $C \in \mathcal{C}_{w+1}$ with $\{0, i_k\} \subset V(C)$ defines exactly the same 5-cycle for T.
- If T meets exactly two fibers, say fibers i_1 and i_2, then consider the vertex triple $\{0, i_1, i_2\}$. It occurs in precisely one $C \in \mathcal{C}_{w+1}$, which defines a bijection φ_C. The inverse mapping φ_C^{-1} determines a vertex triple $\varphi_C^{-1}(T) \subset V(\mathcal{C}_{4v+1})$, which is contained in precisely one 5-cycle $C_T \in \mathcal{C}_{4v+1}$. Then $\varphi(C_T)$ is a 5-cycle containing T in $\mathcal{C}^*$. These steps for T are unambiguous because the derived system of $\mathcal{B}[\mathcal{C}_{w+1}]$ is a $\mathcal{S}(2,4,w)$
- If T meets three fibers, say fibers i_1, i_2 and i_3, then consider the vertex triple $\{i_1, i_2, i_3\}$ in $\mathcal{C}_{w+1}$. It occurs in precisely one $C = (i_1, i_2, i_3, i_4, i_5) \in \mathcal{C}_{w+1}$. Now two situations are possible, depending on whether $0 \in V(C)$ or not. If $0 \in V(C)$, then by assumption the block $B = V(C) \in \mathcal{B}[\mathcal{C}_{w+1}]$ is the unique one containing $\{i_1, i_2, i_3\}$ as a subset, therefore the 5-cycles meeting the three fibers i_1, i_2, i_3 are determined by φ_C. Otherwise, if $0 \notin V(C)$, then we have taken $\mathcal{T}_C$ as a 3-TC$(5, v)$, whose vertex set entirely includes (and consists of) fibers i_1, i_2, i_3, i_4, i_5 minus ∞. Hence there is precisely one cycle in $\mathcal{T}_C$ that contains T, whose 5-cycle specified above has T as an edge.

Thus, $\mathcal{C}^*$ is a $\mathcal{C}(3, 5, vw + 1)$. □

Combining this result with previously known constructions, we can confirm the existence of 3-uniform 5-cycle systems for several new values of the numbers of vertices.

Theorem 5.2 *There exist $\mathcal{C}(3, 5, n)$ systems for each of the orders*

$$n = 97, 145, 151, 161, 241, 251, 266, 361, 385, 401, 577, 601, 641, 901, 1001.$$

Proof. We apply Theorem 5.1 with $4v + 1 = 17, 25, 37, 41$ by Corollary 2.5 and Proposition 2.6, which means $v = 4, 6, 9, 10$, and $w = 17, 26, 41, 65, 101$ by the block-centered system $\mathcal{C}(3, 5, 41)$ of Proposition 3.4 and the known Steiner systems of orders $17, 26, 65, 101$. The details of computation are collected in Table 1. □

$4v+1$	$v+1$	$w+1=$ 17	26	41	65	101
17	5	65	101	161	257	401
25	7	97	151	241	385	601
37	10	145	226	361	577	901
41	11	161	251	401	641	1001

Table 1: Constructions of $\mathcal{C}(3,5,vw+1)$ via Theorem 5.1.

6 Tables of small cycle systems

Here we list the triplet types (see Definition 1.1) generated by the base cycles of systems containing $\mathcal{SP}(4)$ subsystems, on 25, 37, and 41 vertices, respectively. At the end, the detailed description of the block-centered $\mathcal{C}(3,5,41)$ system is also given.

Let us recall from the “Cyclic representation” paragraph of Section 2 that the center x of $\mathcal{SP}(4)$ is fixed and circular symmetry is established in $\mathbb{Z}_{4v}$ for the base cycles whose orbits cover the 3-element sets not contained in the parts of $\mathcal{SP}(4)$. More explicitly, the mapping with $x \mapsto x$ and $i \mapsto i+1$ for all $i \in \mathbb{Z}_{4v}$ will be an automorphism of the subsystem $\mathcal{C}(3,5,4v+1) \setminus \mathcal{SP}4$. Here it is irrelevant whether or not this mapping is an automorphism of the $\mathcal{SP}(4)$ subsystem, too.

In the block-centered system of order 41 (Table 6) the central part consists of two vertices x, y. In this case the other 39 vertices are represented over $\mathbb{Z}_{39}$, and the mapping with $x \mapsto x$, $y \mapsto y$, $i \mapsto i+1$ ($i \in \mathbb{Z}_{39}$) will be an automorphism of the entire system.

The first small table aims to help the interpretation of the data presented in the rest of this section. We take the example base cycle $(0,1,22,2,23)$ from the system $\mathcal{C}(3,5,25)$ and go through the process of determining its triplet types. The first step is to calculate the distances between the vertices of all 5 vertex triples. Recall that the calculation is done in $\mathbb{Z}_{24}$. Within each vertex triple the smallest of the three distances will be denoted by d. The vertex pair with this smallest distance will be the first and second vertex in the ordered version of the triple. The remaining vertex of the original triple will get the third place. The order of the first two vertices will be chosen so that the pair is put in the smaller arc of the cyclic order $0,1,\ldots,v-1,0$.

Table 2: Triplet type calculations explained on the example base cycle $(0, 1, 22, 2, 23)$ from the system $\mathcal{C}(3, 5, 25)$ with an $\mathcal{SP}(4)$ subsystem; computed modulo 24 after removing the center of $\mathcal{SP}(4)$.

Vertices	Distances			d	Direction	Ordered triple	Type
(v_i, v_{i+1}, v_{i+2})	v_i-v_{i+1}	v_{i+1}-v_{i+2}	v_{i+2}-v_i				
(0,1,22)	1	3	2	1	right	0 1 22	1 2 23
(1,22,2)	3	4	1	1	left	1 2 22	1 2 22
(22,2,23)	4	3	1	1	left	22 23 2	1 2 5
(2,23,0)	3	1	2	1	right	23 0 2	1 2 4
(23,0,1)	2	1	1	1	right	23 0 1	1 2 3

Table 3: Base cycles of the system $\mathcal{C}(3, 5, 25)$, modulo 24, without the four $\mathcal{C}(3, 5, 7)$, that form the $\mathcal{SP}(4)$ subsystem.

(v1,v2,v3,v4,v5)	(v1,v2,v3)			(v2,v3,v4)			(v3,v4,v5)			(v4,v5,v1)			(v5,v1,v2)		
	d	triple	type	d	triple	type	d	triple	type	d	triple	type	d	triple	type
Base cycles containing x.															
(x,0,1,13,2)	-	-	-	1	0 1 13	1 2 14	1	1 2 13	1 2 13	-	-	-	-	-	-
(x,0,3,15,6)	-	-	-	3	0 3 15	1 4 16	3	3 6 15	1 4 13	-	-	-	-	-	-
(x,0,5,17,10)	-	-	-	5	0 5 17	1 6 18	5	5 10 17	1 6 13	-	-	-	-	-	-
Base cycles containing the symmetric triplets.															
(0,1,22,2,23)	1	0 1 22	1 2 23	1	1 2 22	1 2 22	1	22 23 2	1 2 5	1	23 0 2	1 2 4	1	23 0 1	1 2 3
(0,2,20,4,22)	2	0 2 20	1 3 21	2	2 4 20	1 3 19	2	20 22 4	1 3 9	2	22 0 4	1 3 7	2	22 0 2	1 3 5
(0,3,19,5,21)	3	0 3 19	1 4 20	2	3 5 19	1 3 17	2	19 21 5	1 3 11	3	21 0 5	1 4 9	3	21 0 3	1 4 7
(0,5,14,10,19)	5	0 5 14	1 6 15	4	10 14 5	1 5 20	4	10 14 19	1 5 10	5	19 0 10	1 6 16	5	19 0 5	1 6 11
(0,6,23,1,18)	1	23 0 6	1 2 8	2	23 1 6	1 3 8	2	23 1 18	1 3 20	1	0 1 18	1 2 19	6	18 0 6	1 7 13
(0,7,10,14,17)	3	7 10 0	1 4 18	3	7 10 14	1 4 8	3	14 17 10	1 4 21	3	14 17 0	1 4 11	7	17 0 7	1 8 15
(0,9,10,14,15)	1	9 10 0	1 2 16	1	9 10 14	1 2 6	1	14 15 10	1 2 21	1	14 15 0	1 2 11	6	9 15 0	1 7 16
(0,10,4,20,14)	4	0 4 10	1 5 11	6	4 10 20	1 7 17	6	14 20 4	1 7 15	4	20 0 14	1 5 19	4	10 14 0	1 5 15
(0,11,2,22,13)	2	0 2 11	1 3 12	4	22 2 11	1 5 14	4	22 2 13	1 5 16	2	22 0 13	1 3 16	2	11 13 0	1 3 14
First half of the reflected cycle pairs.															
(0,11,8,17,1)	3	8 11 0	1 4 17	3	8 11 17	1 4 10	7	1 8 17	1 8 17	1	0 1 17	1 2 18	1	0 1 11	1 2 12
(0,2,12,23,5)	2	0 2 12	1 3 13	3	23 2 12	1 4 14	6	23 5 12	1 7 14	1	23 0 5	1 2 7	2	0 2 5	1 3 6
(0,16,1,18,5)	1	0 1 16	1 2 17	2	16 18 1	1 3 10	4	1 5 18	1 5 18	5	0 5 18	1 6 19	5	0 5 16	1 6 17
Second half of the reflected cycle pairs.															
(0,8,17,14,1)	7	17 0 8	1 8 16	3	14 17 8	1 4 19	3	14 17 1	1 4 12	1	0 1 14	1 2 15	1	0 1 8	1 2 9
(0,6,17,3,5)	6	0 6 17	1 7 18	3	3 6 17	1 4 15	2	3 5 17	1 3 15	2	3 5 0	1 3 22	1	5 6 0	1 2 20
(0,11,4,13,5)	4	0 4 11	1 5 12	2	11 13 4	1 3 18	1	4 5 13	1 2 10	5	0 5 13	1 6 14	5	0 5 11	1 6 12

Table 4: Base cycles of the system $\mathcal{C}(3,5,37)$, modulo 36, without the four $\mathcal{C}(3,5,10)$, that form the $\mathcal{SP}(4)$ subsystem.

(v1,v2,v3,v4,v5)	(v1,v2,v3)			(v2,v3,v4)			(v3,v4,v5)			(v4,v5,v1)			(v5,v1,v2)		
	d	triple	type	d	triple	type	d	triple	type	d	triple	type	d	triple	type
Base cycles containing x. The fourth cycle has only 18 positions.															
(x,0,1,19,2)	-	-	-		0 1 19	1 2 20		1 2 19	1 2 19	-	-	-	-	-	-
(x,0,3,21,6)	-	-	-		0 3 21	1 4 22		3 6 21	1 4 19	-	-	-	-	-	-
(x,0,5,23,10)	-	-	-		0 5 23	1 6 24		5 10 23	1 6 19	-	-	-	-	-	-
(x,0,7,25,14)	-	-	-		0 7 25	1 8 26		7 14 25	1 8 19	-	-	-	-	-	-
(x,0,9,27,18)	-	-	-		0 9 27	1 10 28		18 27 9	1 10 28	-	-	-	-	-	-
Base cycles containing the symmetric triplets.															
(0,1,34,2,35)	1	0 1 34	1 2 35	1	1 2 34	1 2 34	1	34 35 2	1 2 5	1	35 0 2	1 2 4	1	35 0 1	1 2 3
(0,2,32,4,34)	2	0 2 32	1 3 33	2	2 4 32	1 3 31	2	32 34 4	1 3 9	2	34 0 4	1 3 7	2	34 0 2	1 3 5
(0,3,30,6,33)	3	0 3 30	1 4 31	3	3 6 30	1 4 28	3	30 33 6	1 4 13	3	33 0 6	1 4 10	3	33 0 3	1 4 7
(0,5,26,10,31)	5	0 5 26	1 6 27	5	5 10 26	1 6 22	5	26 31 10	1 6 21	5	31 0 10	1 6 16	5	31 0 5	1 6 11
(0,6,25,11,30)	6	0 6 25	1 7 26	5	6 11 25	1 6 20	5	25 30 11	1 6 23	6	30 0 11	1 7 18	6	30 0 6	1 7 13
(0,7,22,14,29)	7	0 7 22	1 8 23	7	7 14 22	1 8 16	7	22 29 14	1 8 29	7	29 0 14	1 8 22	7	29 0 7	1 8 15
(0,10,16,20,26)	6	10 16 0	1 7 27	4	16 20 10	1 5 31	4	16 20 26	1 5 11	6	20 26 0	1 7 17	10	26 0 10	1 11 21
(0,11,14,22,25)	3	11 14 0	1 4 26	3	11 14 22	1 4 12	3	22 25 14	1 4 29	3	22 25 0	1 4 15	11	25 0 11	1 12 23
(0,13,10,26,23)	3	10 13 0	1 4 27	3	10 13 26	1 4 17	3	23 26 10	1 4 24	3	23 26 0	1 4 14	10	13 23 0	1 11 24
(0,14,8,28,22)	6	8 14 0	1 7 29	6	8 14 28	1 7 21	6	22 28 8	1 7 23	6	22 28 0	1 7 15	8	14 22 0	1 9 23
(0,15,6,30,21)	6	0 6 15	1 7 16	9	6 15 30	1 10 25	9	21 30 6	1 10 22	6	30 0 21	1 7 28	6	15 21 0	1 7 22
(0,17,2,34,19)	2	0 2 17	1 3 18	4	34 2 17	1 5 20	4	34 2 19	1 5 22	2	34 0 19	1 3 22	2	17 19 0	1 3 20
First half of the reflected cycle pairs.															
(0,1,5,8,10)	1	0 1 5	1 2 6	3	5 8 1	1 4 33	2	8 10 5	1 3 34	2	8 10 0	1 3 29	1	0 1 10	1 2 11
(0,1,6,8,17)	1	0 1 6	1 2 7	2	6 8 1	1 3 32	2	6 8 17	1 3 12	8	0 8 17	1 9 18	1	0 1 17	1 2 18
(0,1,7,12,15)	1	0 1 7	1 2 8	5	7 12 1	1 6 31	3	12 15 7	1 4 32	3	12 15 0	1 4 25	1	0 1 15	1 2 16
(0,1,8,10,21)	1	0 1 8	1 2 9	2	8 10 1	1 3 30	2	8 10 21	1 3 14	10	0 10 21	1 11 22	1	0 1 21	1 2 22
(0,1,9,14,23)	1	0 1 9	1 2 10	5	9 14 1	1 6 29	5	9 14 23	1 6 15	9	14 23 0	1 10 23	1	0 1 23	1 2 24
(0,1,11,35,24)	1	0 1 11	1 2 12	2	35 1 11	1 3 13	11	24 35 11	1 12 24	1	35 0 24	1 2 26	1	0 1 24	1 2 25
(0,2,14,19,23)	2	0 2 14	1 3 15	5	14 19 2	1 6 25	4	19 23 14	1 5 32	4	19 23 0	1 5 18	2	0 2 23	1 3 24
(0,3,20,32,10)	3	0 3 20	1 4 21	7	32 3 20	1 8 25	10	10 20 32	1 11 23	4	32 0 10	1 5 15	3	0 3 10	1 4 11
(0,4,11,16,27)	4	0 4 11	1 5 12	5	11 16 4	1 6 30	5	11 16 27	1 6 17	9	27 0 16	1 10 26	4	0 4 27	1 5 28
(0,4,22,28,15)	4	0 4 22	1 5 23	6	22 28 4	1 7 19	6	22 28 15	1 7 30	8	28 0 15	1 9 24	4	0 4 15	1 5 16
(0,8,25,2,18)	8	0 8 25	1 9 26	6	2 8 25	1 7 24	7	18 25 2	1 8 21	2	0 2 18	1 3 19	8	0 8 18	1 9 19
(0,9,19,2,16)	9	0 9 19	1 10 20	7	2 9 19	1 8 18	3	16 19 2	1 4 23	2	0 2 16	1 3 17	7	9 16 0	1 8 28
Second half of the reflected cycle pairs.															
(0,2,5,9,10)	2	0 2 5	1 3 6	3	2 5 9	1 4 8	1	9 10 5	1 2 33	1	9 10 0	1 2 28	2	0 2 10	1 3 11
(0,9,11,16,17)	2	9 11 0	1 3 28	2	9 11 16	1 3 8	1	16 17 11	1 2 32	1	16 17 0	1 2 21	8	9 17 0	1 9 28
(0,3,8,14,15)	3	0 3 8	1 4 9	5	3 8 14	1 6 12	1	14 15 8	1 2 31	1	14 15 0	1 2 23	3	0 3 15	1 4 16
(0,11,13,20,21)	2	11 13 0	1 3 26	2	11 13 20	1 3 10	1	20 21 13	1 2 30	1	20 21 0	1 2 17	10	11 21 0	1 11 26
(0,9,14,22,23)	5	9 14 0	1 6 28	5	9 14 22	1 6 14	1	22 23 14	1 2 29	1	22 23 0	1 2 15	9	0 9 23	1 10 24
(0,25,13,23,24)	11	25 0 13	1 12 25	2	23 25 13	1 3 27	1	23 24 13	1 2 27	1	23 24 0	1 2 14	1	24 25 0	1 2 13
(0,4,9,21,23)	4	0 4 9	1 5 10	5	4 9 21	1 6 18	2	21 23 9	1 3 25	2	21 23 0	1 3 16	4	0 4 23	1 5 24
(0,14,26,7,10)	10	26 0 14	1 11 25	7	7 14 26	1 8 20	3	7 10 26	1 4 20	3	7 10 0	1 4 30	4	10 14 0	1 5 27
(0,11,16,23,27)	5	11 16 0	1 6 26	5	11 16 23	1 6 13	4	23 27 16	1 5 30	4	23 27 0	1 5 14	9	27 0 11	1 10 21
(0,23,29,11,15)	6	23 29 0	1 7 14	6	23 29 11	1 7 25	4	11 15 29	1 5 19	4	11 15 0	1 5 26	8	15 23 0	1 9 22
(0,16,29,10,18)	7	29 0 16	1 8 24	6	10 16 29	1 7 20	8	10 18 29	1 9 20	8	10 18 0	1 9 27	2	16 18 0	1 3 21
(0,14,33,7,16)	3	33 0 14	1 4 18	7	7 14 33	1 8 27	9	7 16 33	1 10 27	7	0 7 16	1 8 17	2	14 16 0	1 3 23

Table 5: Base cycles of the system $\mathcal{C}(3,5,41)$, modulo 40, without the four $\mathcal{C}(3,5,16)$, that form the $\mathcal{SP}(4)$ subsystem.

(v1,v2,v3,v4,v5)	(v1,v2,v3)			(v2,v3,v4)			(v3,v4,v5)			(v4,v5,v1)			(v5,v1,v2)		
	d	triple	type	d	triple	type	d	triple	type	d	triple	type	d	triple	type
Base cycles containing x.															
(x,0,1,21,2)	-	-	-		0 1 21	1 2 22		1 2 21	1 2 21	-	-	-	-	-	-
(x,0,3,23,6)	-	-	-		0 3 23	1 4 24		3 6 23	1 4 21	-	-	-	-	-	-
(x,0,5,25,10)	-	-	-		0 5 25	1 6 26		5 10 25	1 6 21	-	-	-	-	-	-
(x,0,7,27,14)	-	-	-		0 7 27	1 8 28		7 14 27	1 8 21	-	-	-	-	-	-
(x,0,9,29,18)	-	-	-		0 9 29	1 10 30		9 18 29	1 10 21	-	-	-	-	-	-
Base cycles containing the symmetric triplets.															
(0,1,38,2,39)	1	0 1 38	1 2 39	1	1 2 38	1 2 38	1	38 39 2	1 2 5	1	39 0 2	1 2 4	1	39 0 1	1 2 3
(0,2,36,4,38)	2	0 2 36	1 3 37	2	2 4 36	1 3 35	2	36 38 4	1 3 9	2	38 0 4	1 3 7	2	38 0 2	1 3 5
(0,3,34,6,37)	3	0 3 34	1 4 35	3	3 6 34	1 4 32	3	34 37 6	1 4 13	3	37 0 6	1 4 10	3	37 0 3	1 4 7
(0,5,32,8,35)	5	0 5 32	1 6 33	3	5 8 32	1 4 28	3	32 35 8	1 4 17	5	35 0 8	1 6 14	5	35 0 5	1 6 11
(0,6,28,12,34)	6	0 6 28	1 7 29	6	6 12 28	1 7 23	6	28 34 12	1 7 25	6	34 0 12	1 7 19	6	34 0 6	1 7 13
(0,7,26,14,33)	7	0 7 26	1 8 27	7	7 14 26	1 8 20	7	26 33 14	1 8 29	7	33 0 14	1 8 22	7	33 0 7	1 8 15
(0,9,22,18,31)	9	0 9 22	1 10 23	4	18 22 9	1 5 32	4	18 22 31	1 5 14	9	31 0 18	1 10 28	9	31 0 9	1 10 19
(0,10,21,19,30)	10	0 10 21	1 11 22	2	19 21 10	1 3 32	2	19 21 30	1 3 12	10	30 0 19	1 11 30	10	30 0 10	1 11 21
(0,11,19,21,29)	8	11 19 0	1 9 30	2	19 21 11	1 3 33	2	19 21 29	1 3 11	8	21 29 0	1 9 20	11	29 0 11	1 12 23
(0,13,14,26,27)	1	13 14 0	1 2 28	1	13 14 26	1 2 14	1	26 27 14	1 2 29	1	26 27 0	1 2 15	13	27 0 13	1 14 27
(0,14,12,28,26)	2	12 14 0	1 3 29	2	12 14 28	1 3 17	2	26 28 12	1 3 27	2	26 28 0	1 3 15	12	14 26 0	1 13 27
(0,15,8,32,25)	7	8 15 0	1 8 33	7	8 15 32	1 8 25	7	25 32 8	1 8 24	7	25 32 0	1 8 16	10	15 25 0	1 11 26
(0,17,6,34,23)	6	0 6 17	1 7 18	11	6 17 34	1 12 29	11	23 34 6	1 12 24	6	34 0 23	1 7 30	6	17 23 0	1 7 24
(0,18,4,36,22)	4	0 4 18	1 5 19	8	36 4 18	1 9 23	8	36 4 22	1 9 27	4	36 0 22	1 5 27	4	18 22 0	1 5 23
(0,19,2,38,21)	2	0 2 19	1 3 20	4	38 2 19	1 5 22	4	38 2 21	1 5 24	2	38 0 21	1 3 24	2	19 21 0	1 3 22
First half of the reflected cycle pairs.															
(0,1,5,8,18)	1	0 1 5	1 2 6	3	5 8 1	1 4 37	3	5 8 18	1 4 14	8	0 8 18	1 9 19	1	0 1 18	1 2 19
(0,1,6,3,11)	1	0 1 6	1 2 7	2	1 3 6	1 3 6	3	3 6 11	1 4 9	3	0 3 11	1 4 12	1	0 1 11	1 2 12
(0,1,7,11,16)	1	0 1 7	1 2 8	4	7 11 1	1 5 35	4	7 11 16	1 5 10	5	11 16 0	1 6 30	1	0 1 16	1 2 17
(0,1,8,3,15)	1	0 1 8	1 2 9	2	1 3 8	1 3 8	5	3 8 15	1 6 13	3	0 3 15	1 4 16	1	0 1 15	1 2 16
(0,1,9,15,22)	1	0 1 9	1 2 10	6	9 15 1	1 7 33	6	9 15 22	1 7 14	7	15 22 0	1 8 26	1	0 1 22	1 2 23
(0,1,10,15,29)	1	0 1 10	1 2 11	5	10 15 1	1 6 32	5	10 15 29	1 6 20	11	29 0 15	1 12 27	1	0 1 29	1 2 30
(0,2,9,12,27)	2	0 2 9	1 3 10	3	9 12 2	1 4 34	3	9 12 27	1 4 19	12	0 12 27	1 13 28	2	0 2 27	1 3 28
(0,2,17,6,20)	2	0 2 17	1 3 18	4	2 6 17	1 5 16	3	17 20 6	1 4 30	6	0 6 20	1 7 21	2	0 2 20	1 3 21
(0,2,18,31,12)	2	0 2 18	1 3 19	11	31 2 18	1 12 28	6	12 18 31	1 7 20	9	31 0 12	1 10 22	2	0 2 12	1 3 13
(0,3,21,10,17)	3	0 3 21	1 4 22	7	3 10 21	1 8 19	4	17 21 10	1 5 34	7	10 17 0	1 8 31	3	0 3 17	1 4 18
(0,4,14,32,19)	4	0 4 14	1 5 15	10	4 14 32	1 11 29	5	14 19 32	1 6 19	8	32 0 19	1 9 28	4	0 4 19	1 5 20
(0,5,22,32,17)	5	0 5 22	1 6 23	10	22 32 5	1 11 24	5	17 22 32	1 6 16	8	32 0 17	1 9 26	5	0 5 17	1 6 18
(0,6,15,38,11)	6	0 6 15	1 7 16	8	38 6 15	1 9 18	4	11 15 38	1 5 28	2	38 0 11	1 3 14	5	6 11 0	1 6 35
(0,6,16,37,21)	6	0 6 16	1 7 17	9	37 6 16	1 10 20	5	16 21 37	1 6 22	3	37 0 21	1 4 25	6	0 6 21	1 7 22
(0,9,25,39,16)	9	0 9 25	1 10 26	10	39 9 25	1 11 27	9	16 25 39	1 10 24	1	39 16	1 2 18	7	9 16	1 8 32
Second half of the reflected cycle pairs.															
(0,10,13,17,18)	3	10 13 0	1 4 31	3	10 13 17	1 4 8	1	17 18 13	1 2 37	1	17 18 0	1 2 24	8	10 18 0	1 9 31
(0,8,5,10,11)	3	5 8 0	1 4 36	2	8 10 5	1 3 38	1	10 11 5	1 2 36	1	10 11 0	1 2 31	3	8 11 0	1 4 33
(0,5,9,15,16)	4	5 9 0	1 5 36	4	5 9 15	1 5 11	1	15 16 9	1 2 35	1	15 16 0	1 2 26	5	0 5 16	1 6 17
(0,12,7,14,15)	5	7 12 0	1 6 34	2	12 14 7	1 3 36	1	14 15 7	1 2 34	1	14 15 0	1 2 27	3	12 15 0	1 4 29
(0,7,13,21,22)	6	7 13 0	1 7 34	6	7 13 21	1 7 15	1	21 22 13	1 2 33	1	21 22 0	1 2 20	7	0 7 22	1 8 23
(0,14,19,28,29)	5	14 19 0	1 6 27	5	14 19 28	1 6 15	1	28 29 19	1 2 32	1	28 29 0	1 2 13	11	29 0 14	1 12 26
(0,15,18,25,27)	3	15 18 0	1 4 26	3	15 18 25	1 4 11	2	25 27 18	1 3 34	2	25 27 0	1 3 16	12	15 27 0	1 13 26
(0,14,3,18,20)	3	0 3 14	1 4 15	4	14 18 3	1 5 30	2	18 20 3	1 3 26	2	18 20 0	1 3 23	6	14 20 0	1 7 27
(0,21,34,10,12)	6	34 0 21	1 7 28	11	10 21 34	1 12 25	2	10 12 34	1 3 25	2	10 12 0	1 3 31	9	12 21 0	1 10 29
(0,7,36,14,17)	4	36 0 7	1 5 12	7	7 14 36	1 8 30	3	14 17 36	1 4 23	3	14 17 0	1 4 27	7	0 7 17	1 8 18
(0,27,5,15,19)	5	0 5 27	1 6 28	10	5 15 27	1 11 23	4	15 19 5	1 5 31	4	15 19 0	1 5 26	8	19 27 0	1 9 22
(0,25,35,12,17)	5	35 0 25	1 6 31	10	25 35 12	1 11 28	5	12 17 35	1 6 24	5	12 17 0	1 6 29	8	17 25 0	1 9 24
(0,13,36,5,11)	4	36 0 13	1 5 18	8	5 13 36	1 9 32	6	5 11 36	1 7 32	5	0 5 11	1 6 12	2	11 13 0	1 3 30
(0,24,5,15,21)	5	0 5 24	1 6 25	9	15 24 5	1 10 31	6	15 21 5	1 7 31	6	15 21 0	1 7 26	3	21 24 0	1 4 20
(0,17,31,7,16)	9	31 0 17	1 10 27	10	7 17 31	1 11 25	9	7 16 31	1 10 25	7	0 7 16	1 8 17	1	16 17 0	1 2 25

Table 6: Block-centered $\mathcal{C}(3,5,41)$ system; types computed modulo 39.

(a) A block containing both x and y. This block has 13 positions.

(v1,v2,v3,v4,v5)	(v1,v2,v3) (v1,v3,v5)			(v2,v3,v4) (v3,v5,v2)			(v3,v4,v5) (v5,v2,v4)			(v4,v5,v1) (v2,v4,v1)			(v5,v1,v2) (v4,v1,v3)		
	d	triple	type	d	triple	type	d	triple	type	d	triple	type	d	triple	type
(x,y,0,13,26)	-	x y 0	-	13	y 0 13	-	13	0 13 26	1 14 27	13	x 13 26	-	-	x y 26	-
	13	x 0 26	-	13	y 0 26	-	13	y 13 26	-	-	x y 13	-	13	x 0 13	-

(b) Blocks containing either x or y.

(v1,v2,v3,v4,v5)	(v1,v2,v3) (v1,v3,v5)			(v2,v3,v4) (v3,v5,v2)			(v3,v4,v5) (v5,v2,v4)			(v4,v5,v1) (v2,v4,v1)			(v5,v1,v2) (v4,v1,v3)		
	d	triple	type	d	triple	type	d	triple	type	d	triple	type	d	triple	type
Blocks containing x only.															
(x,0,1,6,31)	1	x 0 1	-	1	0 1 6	1 2 7	5	1 6 31	1 6 31	14	x 6 31	-	8	x 0 31	-
	9	x 1 31	-	1	0 1 31	1 2 32	6	0 6 31	1 7 32	6	x 0 6	-	5	x 1 6	-
(x,0,2,12,23)	2	x 0 2	-	2	0 2 12	1 3 13	10	2 12 23	1 11 22	11	x 12 23	-	16	x 0 23	-
	18	x 2 23	-	2	0 2 23	1 3 24	11	12 23 0	1 12 28	12	x 0 12	-	10	x 2 12	-
(x,0,3,7,22)	3	x 0 3	-	3	0 3 7	1 4 8	4	3 7 22	1 5 20	15	x 7 22	-	17	x 0 22	-
	19	x 3 22	-	3	0 3 22	1 4 23	7	0 7 22	1 8 23	7	x 0 7	-	4	x 3 7	-
Blocks containing y only.															
(y,0,1,9,34)	1	y 0 1	-	1	0 1 9	1 2 10	6	34 1 9	1 7 15	14	y 9 34	-	5	y 0 34	-
	6	y 1 34	-	1	0 1 34	1 2 35	5	34 0 9	1 6 15	9	y 0 9	-	8	y 1 9	-
(y,0,2,18,29)	2	y 0 2	-	2	0 2 18	1 3 19	11	18 29 2	1 12 24	11	y 18 29	-	10	y 0 29	-
	12	y 2 29	-	2	0 2 29	1 3 30	10	29 0 18	1 11 29	18	y 0 18	-	16	y 2 18	-
(y,0,3,20,35)	3	y 0 3	-	3	0 3 20	1 4 21	7	35 3 20	1 8 25	15	y 20 35	-	4	y 0 35	-
	7	y 3 35	-	3	0 3 35	1 4 36	4	35 0 20	1 5 25	19	y 0 20	-	17	y 3 20	-

(c) Blocks containing symmetric triplets. The symmetric triplets are highlighted.

(v1,v2,v3,v4,v5)	(v1,v2,v3) (v1,v3,v5)			(v2,v3,v4) (v3,v5,v2)			(v3,v4,v5) (v5,v2,v4)			(v4,v5,v1) (v2,v4,v1)			(v5,v1,v2) (v4,v1,v3)		
	d	triple	type	d	triple	type	d	triple	type	d	triple	type	d	triple	type
(0,1,35,4,38)	1	0 1 35	1 2 36	3	1 4 35	1 4 35	3	35 38 4	1 4 9	1	38 0 4	1 2 6	1	38 0 1	1 2 3
	1	38 0 35	1 2 37	2	38 1 35	1 3 37	2	38 1 4	1 3 6	1	0 1 4	1 2 5	4	0 4 35	1 5 9
(0,2,31,8,37)	2	0 2 31	1 3 32	6	2 8 31	1 7 30	6	31 37 8	1 7 17	2	37 0 8	1 3 11	2	37 0 2	1 3 5
	2	37 0 31	1 3 34	4	37 2 31	1 5 34	4	37 2 8	1 5 11	2	0 2 8	1 3 9	8	0 8 31	1 9 17
(0,3,29,10,36)	3	0 3 29	1 4 30	7	3 10 29	1 8 27	7	29 36 10	1 8 21	3	36 0 10	1 4 14	3	36 0 3	1 4 7
	3	36 0 29	1 4 33	6	36 3 29	1 7 33	6	36 3 10	1 7 14	3	0 3 10	1 4 11	10	0 10 29	1 11 21
(0,5,21,18,34)	5	0 5 21	1 6 22	3	18 21 5	1 4 27	3	18 21 34	1 4 17	5	34 0 18	1 6 24	5	34 0 5	1 6 11
	5	34 0 21	1 6 27	10	34 5 21	1 11 27	10	34 5 18	1 11 24	5	0 5 18	1 6 19	3	18 21 0	1 4 22
(0,6,20,19,33)	6	0 6 20	1 7 21	1	19 20 6	1 2 27	1	19 20 33	1 2 15	6	33 0 19	1 7 26	6	33 0 6	1 7 13
	6	33 0 20	1 7 27	12	33 6 20	1 13 27	12	33 6 19	1 13 26	6	0 6 19	1 7 20	1	19 20 0	1 2 21
(0,7,27,12,32)	7	0 7 27	1 8 28	5	7 12 27	1 6 21	5	27 32 12	1 6 25	7	32 0 12	1 8 20	7	32 0 7	1 8 15
	5	27 32 0	1 6 13	5	27 32 7	1 6 20	5	7 12 32	1 6 26	5	7 12 0	1 6 33	12	0 12 27	1 13 25
(0,9,22,17,30)	9	0 9 22	1 10 23	5	17 22 9	1 6 32	5	17 22 30	1 6 14	9	30 0 17	1 10 27	9	30 0 9	1 10 19
	8	22 30 0	1 9 18	8	22 30 9	1 9 27	8	9 17 30	1 9 22	8	9 17 0	1 9 31	5	17 22 0	1 6 23
(0,11,24,15,28)	11	0 11 24	1 12 25	4	11 15 24	1 5 14	4	24 28 15	1 5 31	11	28 0 15	1 12 27	11	28 0 11	1 12 23
	4	24 28 0	1 5 16	4	24 28 11	1 5 27	4	11 15 28	1 5 18	4	11 15 0	1 5 29	9	15 24 0	1 10 25
(0,14,23,16,25)	9	14 23 0	1 10 26	2	14 16 23	1 3 10	2	23 25 16	1 3 33	9	16 25 0	1 10 24	11	14 25 0	1 12 26
	2	23 25 0	1 3 17	2	23 25 14	1 3 31	2	14 16 25	1 3 12	2	14 16 0	1 3 26	7	16 23 0	1 8 24

(d) The remaining triplets are organized into reflected cycle pairs.

(v1,v2,v3,v4,v5)	(v1,v2,v3)			(v2,v3,v4)			(v3,v4,v5)			(v4,v5,v1)			(v5,v1,v2)		
	d	triple	type	d	triple	type	d	triple	type	d	triple	type	d	triple	type
First half of the reflected cycle pairs.															
(0,1,3,7,15)	1	0 1 3	1 2 4	2	1 3 7	1 3 7	4	3 7 15	1 5 13	7	0 7 15	1 8 16	1	0 1 15	1 2 16
(0,1,7,10,18)	1	0 1 7	1 2 8	3	7 10 1	1 4 34	3	7 10 18	1 4 12	8	10 18 0	1 9 30	1	0 1 18	1 2 19
(0,1,8,3,12)	1	0 1 8	1 2 9	2	1 3 8	1 3 8	4	8 12 3	1 5 35	3	0 3 12	1 4 13	1	0 1 12	1 2 13
(0,1,10,17,21)	1	0 1 10	1 2 11	7	10 17 1	1 8 31	4	17 21 10	1 5 33	4	17 21 0	1 5 23	1	0 1 21	1 2 22
(0,1,13,31,11)	1	0 1 13	1 2 14	9	31 1 13	1 10 22	2	11 13 31	1 3 21	8	31 0 11	1 9 20	1	0 1 11	1 2 12
(0,2,14,37,13)	2	0 2 14	1 3 15	4	37 2 14	1 5 17	1	13 14 37	1 2 25	2	37 0 13	1 3 16	2	0 2 13	1 3 14
(0,3,25,32,14)	3	0 3 25	1 4 26	7	25 32 3	1 8 18	7	25 32 14	1 8 29	7	32 0 14	1 8 22	3	0 3 14	1 4 15
(0,4,18,8,23)	4	0 4 18	1 5 19	4	4 8 18	1 5 15	5	18 23 8	1 6 30	8	0 8 23	1 9 24	4	0 4 23	1 5 24
(0,5,17,34,11)	5	0 5 17	1 6 18	10	34 5 17	1 11 23	6	11 17 34	1 7 24	5	34 0 11	1 6 17	5	0 5 11	1 6 12
(0,6,21,3,15)	6	0 6 21	1 7 22	3	3 6 21	1 4 19	6	15 21 3	1 7 28	3	0 3 15	1 4 16	6	0 6 15	1 7 16
(0,8,22,37,20)	8	0 8 22	1 9 23	10	37 8 22	1 11 25	2	20 22 37	1 3 18	2	37 0 20	1 3 23	8	0 8 20	1 9 21
(0,9,20,3,19)	9	0 9 20	1 10 21	6	3 9 20	1 7 18	1	19 20 3	1 2 24	3	0 3 19	1 4 20	9	0 9 19	1 10 20
Second half of the reflected cycle pairs.															
(0,8,12,14,15)	4	8 12 0	1 5 32	2	12 14 8	1 3 36	1	14 15 12	1 2 38	1	14 15 0	1 2 26	7	8 15 0	1 8 32
(0,8,11,17,18)	3	8 11 0	1 4 32	3	8 11 17	1 4 10	1	17 18 11	1 2 34	1	17 18 0	1 2 23	8	0 8 18	1 9 19
(0,9,4,11,12)	4	0 4 9	1 5 10	2	9 11 4	1 3 35	1	11 12 4	1 2 33	1	11 12 0	1 2 29	3	9 12 0	1 4 31
(0,4,11,20,21)	4	0 4 11	1 5 12	7	4 11 20	1 8 17	1	20 21 11	1 2 31	1	20 21 0	1 2 20	4	0 4 21	1 5 22
(0,19,37,10,11)	2	37 0 19	1 3 22	9	10 19 37	1 10 28	1	10 11 37	1 2 28	1	10 11 0	1 2 30	8	11 19 0	1 9 29
(0,15,38,11,13)	1	38 0 15	1 2 17	4	11 15 38	1 5 28	2	11 13 38	1 3 28	2	11 13 0	1 3 29	2	13 15 0	1 3 27
(0,21,28,11,14)	7	21 28 0	1 8 19	7	21 28 11	1 8 30	3	11 14 28	1 4 18	3	11 14 0	1 4 29	7	14 21 0	1 8 26
(0,15,5,19,23)	5	0 5 15	1 6 16	4	15 19 5	1 5 30	4	19 23 5	1 5 26	4	19 23 0	1 5 21	8	15 23 0	1 9 25
(0,16,33,6,11)	6	33 0 16	1 7 23	10	6 16 33	1 11 28	5	6 11 33	1 6 28	5	6 11 0	1 6 34	5	11 16 0	1 6 29
(0,12,33,9,15)	6	33 0 12	1 7 19	3	9 12 33	1 4 25	6	9 15 33	1 7 25	6	9 15 0	1 7 31	3	12 15 0	1 4 28
(0,22,37,12,20)	2	37 0 22	1 3 25	10	12 22 37	1 11 26	8	12 20 37	1 9 26	8	12 20 0	1 9 28	2	20 22 0	1 3 20
(0,16,38,10,19)	1	38 0 16	1 2 18	6	10 16 38	1 7 29	9	10 19 38	1 10 29	9	10 19 0	1 10 30	3	16 19 0	1 4 24

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