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Proper rainbow-cycle-forbidding edge colorings of graphs

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This paper is dedicated to world peace and to the memory of the late Dean Hoffman.

Abstract

It is well known that if the edges of a finite simple graph on n vertices are colored so that no cycle is rainbow, then no more than $n-1$ colors may appear on the edges. In previous work it was shown that a certain structure theorem held true for edge colorings with exactly $n-1$ colors appearing in which no cycle is rainbow. In this paper we consider edge colorings which are proper and "forbid" rainbow cycles, without the condition that the maximum number of colors must appear.

Two questions about the eponymous edge colorings are considered: which graphs have such a coloring, and, for a graph G that does have such a coloring, for what values of k does G have a coloring with exactly k colors appearing?

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1 Introduction

Let G be an edge-colored graph and H be a subgraph with the induced edge coloring. If no color appears more than once on H, then H is said to be **rainbow**. An edge coloring of G is **rainbow-cycle-forbidding (RCF)** if no cycle in G is rainbow with respect to the edge coloring.

Obviously no graph with a loop can have an RCF edge coloring, since a loop with an assigned color is a rainbow cycle. Also, since a double edge is a cycle of length 2, in an RCF coloring of any multigraph, all of the edges joining any particular pair of vertices must bear the same color. Therefore in investigating RCF colorings of graphs we may as well confine our attention to simple graphs. From here on, every graph shall be finite and simple.

It is easy to see that any RCF coloring on a graph with n vertices can bear at most $n-1$ colors. Although the statement is well known and a proof is given in [4], to save the reader some trouble we will supply the proof in the next section. More generally, an edge coloring of a graph G with c components and n vertices cannot be RCF if it uses more than $n-c$ colors. It is shown in [3] (and again we repeat a proof in the next section) that we can always construct an RCF edge coloring in which this maximum number of colors appears. To maintain consistency with the terminology in [1, 3, 4, 5], we shall call an RCF edge coloring on a graph G with c components and n vertices in which exactly $n-c$ colors appear a **JL-coloring** of G. The authors feel it is important to note that in [1], the JL-colorings of complete graphs are characterized. These edge colorings are Gallai colorings, which are edge colorings of complete graphs in which no K_3 is rainbow. It was not known to the authors of [1] that one of their main results was a special case of a characterization of Gallai colorings — see [2].

However, that result in [1] led beyond Gallai colorings, which are simply RCF edge colorings of complete graphs, to the results in [5], [4], and, finally, [3], in which JL-colorings of arbitrary connected (finite, simple) graphs are characterized. The chracterization: given a connected graph G on at least two vertices, each JL-coloring of G is obtained by an instance of the process described in the proof of Proposition 2.2, below; furthermore, each such instance produces a JL-coloring of G.

A coloring of the edges of a graph is **proper** if no two adjacent edges (i.e., edges sharing a vertex) bear the same color. The **chromatic index**, or **edge chromatic number**, of a graph G, denoted by $\chi'(G)$, is the smallest number of colors in a proper edge coloring of G. A proper RCF (or PRCF for short) edge coloring of a graph is an edge coloring which is both proper and rainbow-cycle-forbidding. When we say PRCF coloring we mean a PRCF edge coloring. A PRCF coloring is a particular kind of **mixed hypergraph** coloring in the sense of Voloshin [7]; the literature on this subject has expanded greatly over the few decades since the topic's introduction. In Section 3 we introduce some basic definitions and concepts about PRCF colorings. In Section 4 we suppose a graph G does admit a PRCF coloring and try to determine for which values of k does G have a PRCF coloring where exactly k colors appear. Finally, in Section 5 we discuss minimal graphs that cannot be PRCF colored, in pursuit of a forbidden-induced-subgraph characterization of PRCF-colorability.

2 Preliminary proofs

At least the first of the following fundamental results about RCF colorings is well known. While Proposition 2.2 has appeared in several other papers written by the authors and may be considered well known by readers familiar with the topics, we include a proof for completeness.

Proposition 2.1. *Suppose that G is a connected graph on n vertices with an RCF edge coloring. Then the number of colors appearing on the edges of G is at most $n-1$.*

Proof. Suppose the number of colors appearing on G is greater than or equal to n. Choose n edges of different colors from G and let H be the subgraph of G induced by these edges. Since H has n edges and $|V(H)| \leq n$, then H, and hence G, necessarily contains a rainbow cycle. □

Proposition 2.2. *If G is a connected graph on n vertices, then there is an RCF edge coloring of G in which $n-1$ colors appear.*

Proof. We will describe a general construction for such a coloring, providing more than is necessary in the proposition.

Let T be a spanning tree in G and let the $n-1$ edges of T be colored so that T is rainbow. The $n-1$ colors appearing on T will be the only colors appearing in the RCF coloring of G whose construction will now be described.

Choose any edge $e \in E(T)$. Then $T-e$ is the disjoint union of two trees, $T[R]$ and $T[S]$; here R, S is a partition of $V(T) = V(G)$.

Let $[R,S] = \{f \in E(G)|$ one end of f is in R and the other is in $S\}$. We already have $e \in [R,S]$ colored; let all edges of $[R,S]$ bear the same color as e. At this stage, $E(G[R]) \setminus E(T[R])$ and $E(G[S]) \setminus E(T[S])$ remain uncolored.

Before proceeding, notice that no matter how the coloring of $E(G)$ is completed, no cycle in G with at least one vertex in each of R and S can be rainbow, because such a cycle would contain at least two edges in $[R,S]$.

To proceed: for $X \in \{R,S\}$, $T[X]$ is a spanning tree of $G[X]$ and therefore $G[X]$ is connected. If $|X| = 1$ then $G[X]$ has no edges to color. Otherwise if $|X| > 1$, treat $G[X]$ as G was just treated, with the rainbow spanning tree $T[X]$ playing the role played by T. Iterate until all the edges of G are colored. □

Note that in the last part of the preceding proof, the reader can replace $T[X]$ with any other spanning tree in $G[X]$, supplied with the colors on $T[X]$.

Corollary 2.3. *If a graph G has c components, then the greatest number of colors appearing in an RCF edge coloring of G is $|V(G)| - c$.*

3 Proper RCF edge colorings

In a **mixed hypergraph** coloring problem, one has a two-ply hypergraph $\mathcal{H} = (V; \mathcal{C}, \mathcal{F})$ in which V is the set of vertices of $\mathcal{H}$ and $\mathcal{C}, \mathcal{F} \subseteq 2^V$ are two families of **hyperedges** (subsets of V). The problem is to color V so that no $C \in \mathcal{C}$ is rainbow and no $F \in \mathcal{F}$ is monochromatic. We will call such a coloring of V a **proper $(\mathcal{C}, \mathcal{F})$** coloring of (the vertices of) $\mathcal{H}$.

A PRCF coloring is a mixed hypergraph coloring in which the vertices of the hypergraph are the edges of some graph G, and

$$\begin{aligned} \mathcal{C} &= \{E(C) \mid C \text{ is a cycle in } G\} \text{ and} \\ \mathcal{F} &= \{E(F) \mid F \simeq K_{1,2} \text{ and } F \text{ is a subgraph of } G\}. \end{aligned}$$

The standard object of attention in a mixed hypergraph is the **Voloshin spectrum** [7], the set of positive integers k such that there is a proper $(\mathcal{C}, \mathcal{F})$ coloring of the vertices of $\mathcal{H}$ with exactly k colors appearing. We define

$$SPRCF(G) = \{k \in \mathbb{N} \setminus \{0\} \mid \text{there is a PRCF edge coloring of } G \text{ with exactly } k \text{ colors appearing}\}.$$

By Proposition 2.1 and previous remarks, for a connected graph G on n vertices,

$$\begin{aligned} SPRCF(G) &\subseteq \{\chi'(G), \ldots, n-1\} \\ &= \{k \in \mathbb{N} \setminus \{0\} \mid \chi'(G) \leq k \leq n-1\}. \end{aligned}$$

We continue with a few definitions. A graph G is PRCF-**good** if $SPRCF(G) \neq \emptyset$. Otherwise, G is PRCF-**bad**. We will also define the graph with no edges to be PRCF-good as that simplifies some later statements.

If $SPRCF(G)$ is either a singleton or a block of consecutive integers, then G is PRCF-**solid**. G is PRCF-**excellent** if $\chi'(H) < |V(H)|$ for every component of H of G and, with c denoting the number of components of G, $SPRCF(G) = \{\chi'(G), \ldots, |V(G)| - c\}$.

We will note here that not all graphs which are PRCF-solid will be PRCF-excellent. In Theorem 4.1.3 we show that $SPRCF(K_{2,3}) = \{3\}$ as opposed to $\{3, 4\}$ which would be required for PRCF-excellence. We have not found a graph which is PRCF-good but not PRCF-solid. In fact, we strongly suspect that every PRCF-good graph is solid, i.e., we suspect that we do not get any "gaps" in the spectrum, except at the end or potentially the beginning of the interval.

Further, we have not even found an example of a graph where $a \in SPRCF(G)$ but $\{\chi'(G), \ldots, a\} \not\subseteq SPRCF(G)$. It would be nice if one could find such an example (or alternatively, show that if $a \in SPRCF(G)$ then $\{\chi'(G), \ldots, a\} \subseteq SPRCF(G)$ if such a thing ends up being true). While we believe that this previous statement would be difficult to prove (if even true), it does bring to mind the following related question. If G is properly JL-colorable (i.e., $|V(G)| - 1 \in SPRCF(G)$), then is G PRCF-excellent? We know that every JL-coloring of a graph has a monochromatic edge-cut

[3] and further, since the coloring is proper, this implies the cut is a matching. Our work in Section 4.6 then goes a long way towards this goal.

Our aim in requiring $\chi'(H) < |V(H)|$ for every component of H in G, in the definition of PRCF-excellence, is to exclude from the class of PRCF-excellent graphs certain graphs which would otherwise satisfy the definition, but which are also PRCF-bad. If $n = |V(G)| > 1$, n is odd, and G is K_n with no more than $\frac{n-3}{2}$ edges removed, then $\chi'(G) = n$, so $\{k \in \mathbb{N} \mid \chi'(G) \leq k \leq |V(G)| - 1\} = \emptyset = SPRCF(G)$.

We shall not prove it here, but it is well known (see Plantholt [6]) that the only connected PRCF-bad graphs G excluded from PRCF-excellence by the provision that $\chi'(G) = |V(G)|$ are those mentioned above.

There is another reason besides $|\chi'| = |V(G)|$ for the PRCF-badness of these graphs: they contain K_3's. In any proper edge coloring of any graph G, any K_3 subgraph of G must be rainbow. Therefore, every PRCF-good graph must be triangle-free.

As we shall see, PRCF-goodness is rare even among triangle-free graphs. The following proposition is trivial to prove, yet it is of considerable importance.

Proposition 3.1. *Every subgraph of a PRCF-good graph is PRCF-good.*

Proof. Suppose G has a PRCF edge coloring, and H is a subgraph of G. The restriction of the coloring to the edges of H is proper, and also forbids rainbow cycles, since any cycle in H is a cycle in G. Therefore, H has a PRCF edge coloring, and is therefore PRCF-good. □

A **forbidden-subgraph characterization** of a property Q of graphs is a collection $\mathcal{F}$ of graphs such that a graph G has property Q if and only if G has no subgraph $F \in \mathcal{F}$. A **forbidden-induced-subgraph characterization** of Q is a collection $\mathcal{F}$ of graphs such that a graph G has property Q if and only if G has no induced subgraph $F \in \mathcal{F}$. A property Q has a forbidden(-induced)-subgraph characterization if and only if every (induced) subgraph of a graph with property Q has property Q. When Q has such a characterization, the minimal forbidden-subgraph characterizing collection for Q is

$$\mathcal{F}_Q = \{H \mid H \text{ is a graph not possessing property } Q\text{, but every proper subgraph of } H \text{ does possess property } Q\}.$$

The minimal forbidden-induced-subgraph characterizing collection of Q, when there is one, is

$$\mathcal{F}_{Q,i} = \{H \mid H \text{ is a graph not possessing property } Q\text{, but } H - v \text{ does possess property } Q\text{, for each } v \in V(H)\}.$$

Corollary 3.2. *The property of being PRCF-good has a forbidden-subgraph characterization and a forbidden-induced-subgraph characterization.*

A graph H is **critically** PRCF-bad if and only if H is PRCF-bad but every proper subgraph of H is PRCF-good. H is **vertex-critically** PRCF-bad if and only if H is PRCF-bad but $H - v$ is PRCF-good for every $v \in V(H)$.

Proposition 3.3. *Suppose G is a graph. The following are equivalent.*

1. *G is PRCF-good.*
2. *G has no critically PRCF-bad subgraph.*
3. *G has no vertex-critically PRCF-bad induced subgraph.*

Proof. If G is PRCF-good, then by Proposition 3.1, every subgraph is PRCF-good and thus cannot have a critically PRCF-bad subgraph nor a vertex-critically PRCF-bad subgraph. Therefore, 1 immediately implies both 2 and 3.

To show 3 implies 1, we suppose G is PRCF-bad and show that G has a vertex-critically PRCF-bad induced subgraph. If no $v \in V(G)$ can be found such that $G - v$ is PRCF-bad, then G itself is vertex-critically PRCF-bad, and is an induced subgraph of itself.

Otherwise, suppose $G_1 = G - v_1$ is PRCF-bad for some $v_1 \in V(G)$ and note that G_1 is an induced subgraph of G. We continue to remove vertices from G to obtain a sequence of vertices $v_1, \dots, v_k \in V(G)$ and induced subgraphs $G_k = G - \{v_1, \dots, v_k\}$ which are PRCF-bad. Since we start with a finite graph, this process eventually leads us to a graph with three vertices (otherwise we would have found a vertex-critically PRCF-bad induced subgraph). The only PRCF-bad subgraph on three vertices is K_3, and removing any vertex leaves a PRCF-good, and thus K_3 is our vertex-critically PRCF-bad induced subgraph.

The proof that 2 implies 1 is similar. Assuming again that G is PRCF-bad, we remove edges to obtain PRCF-bad subgraphs of G, taking care to remove vertices that have become isolated, until no more such edges can be found. As before, since we have a finite graph, this process must reach a graph with three vertices (if we have not yet found a critically PRCF-bad subgraph). The only PRCF-bad subgraph on three vertices is K_3 and removing any edge leaves a PRCF-good graph, so G has a critically PRCF-bad subgraph. □

Lemma 3.4. *A graph H is critically PRCF-bad if and only if H is connected, PRCF-bad, and $H - e$ is PRCF-good for every $e \in E(H)$.*

Proof. This follows immediately from the definition of a graph H being critically PRCF-bad and Proposition 3.1. □

The next section will contain results on the PRCF spectra of graphs. The last section is devoted to critically PRCF-bad graphs. We currently do not have a concise description of these graphs, but at least we have ways of producing two infinite families of critically PRCF-bad graphs.

4 PRCF spectra

If $a, b \in \mathbb{N}$ and $a \le b$ then we will use the notation $[a, b] = \{k \in \mathbb{N} \mid a \le k \le b\}$. If $a_1, \dots, a_t \in \mathbb{N}$, then we also define $s(a_1, \dots, a_t) = [\max\{a_1, \dots, a_t\}, \sum_{i=1}^{t} a_i]$. We will denote the disjoint union of graphs $H_1, \dots, H_t$ by $H_1 + \cdots + H_t$.

4.1 Trees, cycles, and complete bipartite graphs

Theorem 4.1.1. *Every tree is PRCF-excellent.*

Proof. Let T be a tree. Since T has no cycles, a coloring of $E(T)$ is a PRCF coloring if and only if it is a proper edge coloring. Therefore $\chi'(T) \in SPRCF(T)$. From a proper edge coloring of T with $\chi'(T)$ colors appearing we can obtain proper edge colorings with $\chi'(T)+1, \chi'(T)+2, \ldots, |E(T)| = |V(T)|-1$ colors appearing by, at each stage, recoloring one edge representing a color class with more than one representative with a new color not previously appearing. □

We note here that a second proof follows from Corollary 4.4.5, which we will give in Subsection 4.4.

Theorem 4.1.2. *Every cycle of order $n \geq 4$ is PRCF-excellent.*

Proof. Suppose that C is a cycle on $n \geq 4$ vertices. Since $\chi'(C) \in \{2, 3\}$, in any proper edge coloring of C with $\chi'(C)$ colors, C will not be rainbow. Thus $\chi'(C) \in SPRCF(C)$. Choose two edges with the same color in some proper edge coloring of C with $\chi'(C)$ colors. Recolor one of these edges with a new color, not already appearing. Continue in this way, recoloring one edge at a time from a color class with at least 2 representatives with a new color, until there are $n-1$ colors appearing (so only one color appears twice). At each stage, the new coloring is proper and the cycle is still not rainbow. □

We note here that again we have a second proof. The previous theorem also follows from Corollary 4.6.2, but as that proof is not shorter than the one already given, we will leave those details as an exercise for the reader.

Theorem 4.1.3. $SPRCF(K_{2,3}) = \{3\}$*. Furthermore, there is essentially only one way to PRCF color the edges of* $K_{2,3}$ *with exactly three colors appearing.*

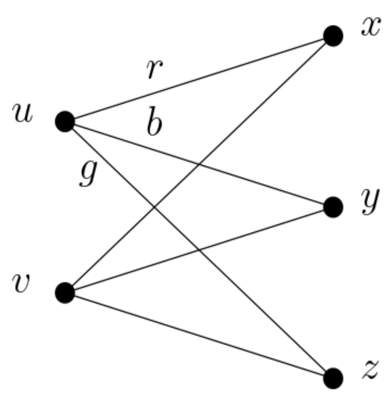


Figure 1: $K_{2,3}$ with the beginning of a PRCF edge coloring.

Proof. Let the three colors with which the edges of $K_{2,3}$ will be colored be r, b, g, and let the vertices of $K_{2,3}$ be labeled as in Figure 1. Without loss of generality, we color the three edges incident to u as shown.

The 4-cycle $uyvxu$ must be properly colored with at least one color repeated. Since there are only three colors allowed, it must be either that vy will be colored r, or vx will be colored b. If we choose one of the possibilities and interchange the names of x and y, and of r and b, we obtain the other choice. Therefore, without loss of generality, we color vx with b. Looking at $uxvzu$, we see that this choice forces us to color vz with r and then vy with g; the result is a PRCF coloring. This coloring is unique up to automorphisms of $K_{2,3}$ and renaming of the colors.

Since $SPRCF(K_{2,3}) \subseteq \{3, 4\}$, to finish the proof it suffices to show that there is no PRCF coloring of $K_{2,3}$ with four colors appearing. Let the available colors be a, r, b, g and, without loss of generality, let us begin with the coloring of the edges incident to u shown in Figure 1. Even with a fourth color at our disposal, we still have two equivalent choices: either vy will be colored r, or vx will be colored b. So, we choose to color vx with b. Only vy and vz remain to be colored, and a has not yet appeared. Putting color a on vz creates a rainbow cycle. If vy is colored a, then propriety forces us to color vz with r, and that creates a rainbow cycle. □

Corollary 4.1.4.

1. *$K_{2,4}$ is critically PRCF-bad.*

2. *$SPRCF(K_{3,3}) = \{3\}$.*

Proof.

1. Let $G = K_{2,4}$, with vertices as labeled in Figure 2. Since $\chi'(G) = 4$ and $|V(G)| = 6$, we have $SPRCF(G) \subseteq \{4, 5\}$. Suppose $K_{2,4}$ has a PRCF-coloring. Then the $K_{2,3}$ subgraph $G - w$ is colored with only 3 of these colors, say r, b, and g, and by Theorem 4.1.3 and its proof, we can assume, without loss of generality, that it is colored as indicated in Figure 2. Since all three colors appear at both u and v, and since the two edges incident to w must be colored differently, the only way to complete the coloring of $G - w$ to a proper coloring of G is to put a fourth and fifth color on the two edges incident to w. But that creates 3 rainbow C_4's. Thus $SPRCF(K_{2,4}) = \emptyset$.

   On the other hand, if one edge, say wu, is removed from G, clearly there is a PRCF coloring of the graph remaining. (In fact, although it is not claimed, we see that for any edge $e \in E(G), SPRCF(G - e) = \{4\}$, and there is an essentially unique PRCF coloring of $G - e$ with four colors.) Thus $K_{2,4}$ is critically PRCF-bad.

2. We leave it to the reader to see that $K_{3,3}$ has a PRCF coloring with three colors appearing. (In fact, $K_{3,3}$ has essentially only one such coloring.) Now suppose that $K_{3,3}$ is PRCF-colored. Let the bipartition sets of $K_{3,3}$ be $\{u, v, w\}$ and $\{x, y, z\}$. The $K_{2,3}$ induced by $\{u, v, x, y, z\}$ bears only three colors, and all 3

appear on the edges incident to u, and incident to v. Therefore, these same three colors are the only colors appearing on any $K_{2,3}$ in $K_{3,3}$; consequently, no other color appears. Thus $SPRCF(K_{3,3}) = \{3\}$. □

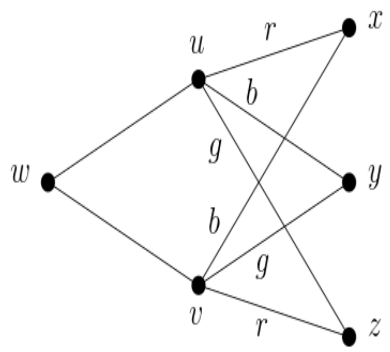


Figure 2: $K_{2,4}$ with a PRCF coloring of one of its $K_{2,3}$ subgraphs.

Corollary 4.1.5. *The only complete bipartite graphs which are PRCF-good are* $K_{2,3}$*,* $K_{3,3}$*, and* $K_{1,b}$ *for* $b \in \{1, 2, \dots\}$.

Proof. The $K_{1,b}$ are trees, and therefore are PRCF-excellent. If $2 \leq a$ and $4 \leq b$ then $K_{a,b}$ contains $K_{2,4}$ as a subgraph, and is therefore PRCF-bad. □

4.2 1-subdivisions

The 1-subdivision SG of a graph G is obtained from G by inserting a "new" vertex (of degree 2) into every edge of G. In other words, SG is the result of replacing each edge of G by a path of length 2. In yet other words, $V(SG) = V(G) \cup E(G)$, each edge $uv \in E(G)$ (which we now think of as a vertex in SG) is adjacent to $u, v \in V(G)$, and these are the only adjacencies of elements of $V(SG)$. In an attempt to avoid confusion, for each edge $uv \in E(G)$ we will denote the corresponding vertex of SG by $[uv]$, while vertices of G will retain their names. Thus if $uv \in E(G)$, then $N_{SG}([uv]) = \{u, v\}$, and for $u \in V(G), N_{SG}(u) = \{[uv] | v \in N_G(u)\}$. (Recall that we are considering only simple graphs in this paper.)

Useful facts about SG when G is a finite simple graph:

1. $|V(SG)| = |V(G)| + |E(G)|$ and $|E(SG)| = 2|E(G)|$.
2. SG is bipartite. Therefore, $\chi'(SG) = \Delta(SG)$.
3. $\Delta(G) = \Delta(SG)$ if $\Delta(G) > 1$.

Lemma 4.2.1. *If* G *is connected and* $|V(G)| > 1$ *then at least two vertices of* G *are not cut-vertices.*

Proof. Any vertex which is an end-vertex of a longest path in G cannot be a cut-vertex of G. □

Corollary 4.2.2. *If G is connected and $|V(G)| > 2$ then there exist vertices $v \in V(G)$ such that $G - v$ has no isolated vertices.*

Theorem 4.2.3. *If G has no isolated vertices then $|V(G)| \in SPRCF(SG)$. If, in addition, there is at least one vertex $v \in V(G)$ such that $G - v$ has no isolated vertices, then $|V(G)| - 1 \in SPRCF(SG)$.*

Proof. Supposing G has no isolated vertices, we will color the edges of SG with the elements of $V = V(G)$ as follows: for each $uv \in E(G)$, color $u[uv]$ with v and color $[uv]v$ with u. It is straightforward to see that this coloring is proper, with $|V|$ colors appearing, and that any color appearing on a cycle in SG must appear twice on the cycle, so the coloring is a PRCF coloring of SG. Notice that for each $v \in V$, v does not appear as a color on any edge of SG incident to v.

Now suppose that $w \in V$ and $G' = G - w$ has no isolated vertices. Let the edges of SG' be colored with the elements of $V(G')$, as described above. Extend this coloring to the edges of SG by coloring $u[uw]$ with u for each $u \in N_G(w)$, and then coloring the edges $[uw]w$ with elements of $V(G')$ so that the resulting coloring is proper. Specifically, if $N_G(w) = \{u\}$, then because $G - w$ has no isolated vertices, $V(G')$ must contain vertices other than u. Color $[uw]w$ with one of these. If $d_G(w) > 1$, the edges $[uw]w$ can be colored with the elements of $N_G(w)$, permuted so that $[uw]w$ is not colored with u, for each $u \in N_G(w)$.

Clearly the coloring described is proper. By the earlier part of this proof, there are no rainbow cycles in SG'. The only cycles in SG that are not in SG' must contain w, and none of these are rainbow since every path of length 4 starting at w contains two edges of the same color (namely, u when $w[uw]$ is the first edge of the path). Thus $|V(G')| = |V(G)| - 1 \in SPRCF(SG)$. □

Theorem 4.2.4. *If $n = |V(G)| > 2$ and G is connected, then $[n-1, n+|E(G)|-1] \subseteq SPRCF(SG)$.*

Proof. If $n = 3$ then SG is either P_5 or C_6, so the result follows from what we already know about trees and cycles. Assume that $n > 3$. We will proceed by induction on n.

By Theorem 4.2.3 and Corollary 4.2.2, $\{n-1, n\} \subseteq SPRCF(SG)$. It remains to be shown that $[n+1, n+|E(G)|-1] \subseteq SPRCF(SG)$. By Lemma 4.2.1 there is a vertex $v \in V(G)$ such that $G' = G - v$ is connected. By the induction hypothesis,

$$[n-1+1, n-1+|E(G')|-1] = [n, n+|E(G)| - d_G(v) - 2] \subseteq SPRCF(SG').$$

Suppose that $n+1 \le k \le n+|E(G)|-1$. If $k \le n+|E(G)|-d_G(v)-1$, let $t = k-1$. Otherwise, if $n+|E(G)|-d_G(v) \le k \le n+|E(G)|-1$, let $t = n+|E(G)|-d_G(v)-2$. In either case, $t \in SPRCF(SG')$ and $1 \le k-t \le d_G(v)+1$.

Let SG' be PRCF-colored with exactly t colors appearing. Extend this coloring of SG' to SG as follows: for each $u \in N_G(v)$, color $u[uv]$ with c, a color which is not one of the t colors on SG'. Then color the $d_G(v)$ edges $[uv]v$ with $d_G(v)$ different colors, $k-t-1$ of them *not* among the $t+1$ colors already appearing, and the rest, $d_G(v) - (k-t-1) \in [1, d_G(v)]$ colors drawn from the stock of t colors on SG'.

The result is a proper edge coloring with exactly k colors appearing. There are no rainbow cycles on SG'. Any cycle in SG which is not in SG' must contain v, and any cycle containing v has at least two edges colored c. □

Corollary 4.2.5. *If G is connected, with $|V(G)| > 1$, then SG has a proper JL-coloring.*

Proof. This is obviously true when $|V(G)| = 2$, and for $|V(G)| > 2$ it follows from Theorem 4.2.4 and the fact that $|V(SG)| = |V(G)| + |E(G)|$. □

Corollary 4.2.6. *If G is connected, $n = |V(G)| > 1$, and $\Delta(G) = n - 1$, then SG is PRCF-excellent.*

Proof. This is obvious when $|V(G)| = 2$ and in the other cases it follows from Theorem 4.2.4 and $\chi'(SG) = \Delta(G)$. □

We do not know of any connected graph G with $|V(G)| > 1$ such that SG is not PRCF-excellent.

4.3 Cartesian Products

The Cartesian product of graphs G and H is denoted by $G \,\square\, H$, and is defined by $V(G \,\square\, H) = V(G) \times V(H)$, with ordered pairs $(u, v), (u', v')$ adjacent in $G \,\square\, H$ if and only if either $u = u'$ and $vv' \in E(H)$ or $uu' \in E(G)$ and $v = v'$. There are $|V(G)|$ subgraphs of $G \,\square\, H$ induced by the vertex sets $\{u\} \times V(H), u \in V(G)$, which are all copies of H and there are $|V(H)|$ subgraphs induced by the vertex sets $V(G) \times \{v\}, v \in V(H)$, which are all copies of G. Further, the edge sets of these "coordinate copies" of H and G partition the edge set of $G \,\square\, H$.

We first make a rather obvious claim about sets, the use of which will be apparent in the resultant theorem and later in Subsection 4.5.

Lemma 4.3.1. *Suppose that $a_1, \ldots, a_t \in \mathbb{N}$ and let*

$$s(a_1, \ldots, a_t) = [\max\{a_1, \ldots, a_t\}, \sum_{i=1}^{t} a_i].$$

For each $s \in s(a_1, \ldots, a_t)$ there exists sets $A_1, \ldots, A_t$ such that $|A_i| = a_i$, $i = 1, \ldots, t$ and $\left|\bigcup_{i=1}^{t} A_i\right| = s$.

Proof. The proof will be informal. We can let $a_1 \leq \cdots \leq a_t$. We find the desired sets $A_1, \ldots, A_t$ satisfying $\left|\bigcup_{i=1}^{t} A_i\right| = a_t$ by arranging $A_i \subseteq A_t$ for $i < t$, and we can get $\left|\bigcup_{i=1}^{t} A_i\right| = \sum_{i=1}^{t} a_i$ by taking $A_1, \ldots, A_t$ to be disjoint. Clearly we can get every integer $\left|\bigcup_{i=1}^{t} A_i\right| = s \in (a_t, \sum_{i=1}^{t} a_i)$ by managing the intersections of the sets $A_1, \ldots, A_t$. We leave the formalities to the reader. □

Theorem 4.3.2. *Suppose that G and H are PRCF-good, $n = |V(H)|, a_1, \dots, a_n \in SPRCF(G)$, and $b \in SPRCF(H)$. Then $b + s(a_1, \dots, a_n) \subseteq SPRCF(G \,\square\, H)$.*

Proof. Let the vertices of H be $v_1, \dots, v_n$. Let $A_1, \dots, A_n$ be sets of colors such that $|A_j| = a_j, j = 1, \dots, n$. PRCF-color the edges of the copy of G induced by the vertex set $V(G) \times \{v_j\}$ with the elements of $A_j, j = 1, \dots, n$.

Let B be a set of colors such that $|B| = b$. Let H be PRCF-colored with the elements of B and give each copy of H induced by $\{u\} \times V(H), u \in V(G)$, the same coloring. That is, for each $u \in V(G)$ and edge $vv' \in E(H)$, color the edge $(u, v), (u, v')$ of $G \,\square\, H$ with the color on vv'.

We take B to be disjoint from $\bigcup_{j=1}^{n} A_j$. We now have $G \,\square\, H$ edge-colored with $b + \left|\bigcup_{j=1}^{n} A_j\right|$. colors. Since the coloring is proper in each component graph and B is disjoint from $\bigcup_{j=1}^{n} A_j$, the coloring is proper. There are no rainbow cycles in any subgraph $u \,\square\, H, u \in V(G)$, nor in $G \,\square\, v, v \in V(H)$. Therefore, the only cycles in $G \,\square\, H$ that might be rainbow are those such that neither the first nor the second component of the vertices on the cycle is constant. Let C be such a cycle.

Looking at the second coordinates of the vertices on C, we see that they must be the vertices encountered on a non-trivial closed walk W in H. Because we have arranged that for every $vv' \in E(H)$ and $u, u' \in V(G)$, the edges $(u, v), (u, v')$ and $(u', v), (u', v')$ of $G \,\square\, H$ will bear the same color, it follows that for different traversals of the same edge in W, the different edges of C corresponding to the different traversals of that edge on W must bear the same color. Therefore if there are any re-traversals of edges on W, then C is not rainbow. If there are no such re-traversals, then W must contain a cycle in H, on which some color appears on two different edges. Again, C is not rainbow.

Thus $b + |A_1 \cup \dots \cup A_n| \in SPRCF(G \,\square\, H)$. Since $A_1, \dots, A_n$ were arbitrary, by Lemma 4.3.1 we have $b + s(a_1, \dots, a_n) \subseteq SPRCF(G \,\square\, H)$. □

Corollary 4.3.3. *If G and H have proper JL-colorings, then so does $G \,\square\, H$.*

Proof. If G and H have proper JL-colorings then in Theorem 4.3.2 we can take $b = |V(H)| - 1$ and $a_1 = \dots = a_n = |V(G)| - 1$. Then

$$a_1 + \dots + a_n = n(|V(G)| - 1) = |V(H)|(|V(G)| - 1) \in s(a_1, \dots, a_n) \implies$$

$$\begin{aligned} b + (a_1 + \dots + a_n) &= |V(H)| - 1 + |V(H)|(|V(G)| - 1) \\ &= |V(G)||V(H)| - 1 = |V(G \,\square\, H)| - 1 \\ &\in SPRCF(G \,\square\, H). \end{aligned}$$

So, there is a proper JL-coloring of $G \,\square\, H$. □

Corollary 4.3.4. *If c_1, d_1, c_2, d_2 are positive integers such that $c_1 \le d_1$, $c_2 \le d_2$, $[c_1, d_1] \subseteq SPRCF(G)$, and $[c_2, d_2] \subseteq SPRCF(H)$, then*

$$[c_1 + c_2, \max(d_2 + |V(H)|d_1,\ d_1 + |(V(G)|d_2)] \subseteq SPRCF(G \,\square\, H).$$

Proof. Let $m = |V(G)|$, $n = |V(H)|$. With reference to Theorem 4.3.2, if we take $a_1 = \cdots = a_n = a \in SPRCF(G)$ and any $b \in SPRCF(H)$, we get

$$I(a,b) = b + s(a,\ldots,a) = [a+b, b+na] \subseteq SPRCF(G \,\square\, H).$$

Clearly,

$$\bigcup_{\substack{a\in[c_1,d_1]\\ b\in[c_2,d_2]}} I(a,b) = [c_1+c_2, d_2+nd_1] \subseteq SPRCF(G \,\square\, H).$$

Interchanging the roles of G and H gives $[c_1+c_2, d_1+md_2] \subseteq SPRCF(G \,\square\, H)$, whence
$[c_1+c_2, d_2+nd_1] \cup [c_1+c_2, d_1+md_2] = [c_1+c_2, \max(d_2+nd_1,\ d_1+md_2)] \subseteq SPRCF(G \,\square\, H)$. □

Corollary 4.3.5. *If G and H are PRCF-excellent and $\chi'(G \,\square\, H) = \chi'(G) + \chi'(H)$ then $G \,\square\, H$ is PRCF-excellent.*

Proof. This follows from Corollary 4.3.4, taking $c_1 = \chi'(G)$, $c_2 = \chi'(H)$, $d_1 = |V(G)| - 1$, $d_2 = |V(H)| - 1$. Note the proof of Corollary 4.3.3. □

Corollary 4.3.6. *If G and H are PRCF-excellent, $\chi'(G) = \Delta(G)$, and $\chi'(H) = \Delta(H)$, then $G \,\square\, H$ is PRCF-excellent.*

Proof. Properly coloring the edges of the coordinate copies of G and H in $G \,\square\, H$ with disjoint sets of $\chi'(G)$ and $\chi'(H)$ colors, respectively, we see that $\chi'(G \,\square\, H) \le \chi'(G)+\chi'(H)$ for all finite simple subgraphs G and H. On the other hand $\chi'(G \square H) \ge \Delta(G \,\square\, H) = \Delta(G) + \Delta(H)$. Therefore, if $\chi'(X) = \Delta(X)$, for $X \in \{G, H\}$, then $\chi'(G \,\square\, H) = \chi'(G) + \chi'(H)$, and the result follows from the previous corollary. □

We do not know of any PRCF-excellent graphs G and H such that $G \,\square\, H$ is *not* PRCF-excellent.

4.4 PRCF spectrum of a union of two graphs with one vertex in common

If G is a graph and $v \in V(G)$, let the degree of v in G be denoted by $d_G(v)$.

Theorem 4.4.1. *Suppose that G and H are graphs with $V(G) \cap V(H) = \{v\}$ and $d_G(v), d_H(v) > 0$. If $a \in SPRCF(G)$ and $b \in SPRCF(H)$, then $[\max\{a, b, d_G(v) + d_H(v)\}, a+b] \subseteq SPRCF(G \cup H)$.*

Proof. First, observe that $d_G(v) \le \chi'(G) \le a$ and $d_H(v) \le \chi'(H) \le b$, so $d_G(v) + d_H(v) \le a+b$, with equality if and only if $d_G(v) = \chi'(G) = a$ and $d_H(v) = \chi'(H) = b$.

Let A be the set of colors appearing on a PRCF coloring of G with $|A| = a$ and B be the set of colors appearing on a PRCF coloring H with $|B| = b$. Let C be the set of colors appearing on G incident to v and D be the set of colors appearing on H incident to v.

Since any PRCF-coloring of $G \cup H$ is proper, $C \cap D = \emptyset$, and $|C| = d_G(v)$, $|D| = d_H(v)$. We will start with $A \cap B = \emptyset$ and vary $|A \cup B|$ by renaming the colors in $(A \setminus C) \cup (B \setminus D)$. The sets C and D will remain constant. In the interest of simplicity, we will continue to refer to the varying sets A, B as A and B; $|A| = a$ and $|B| = b$ will remain constant. It is $|A \cup B|$ that will trend down.

Clearly if $A \cap B = \emptyset$ then we get $a+b \in SPRCF(G \cup H)$. With careful renaming of colors, one color at a time, we will reach our goal of $M = \max\{a, b, d_G(v) + d_H(v)\} \in SPRCF(G \cup H)$. For clarity, when we say "rename" a color $q \in B \setminus D$ as a color $r \in C$, we take every instance of the color q and label that edge with the color r. This decreases the color count by 1, keeps the coloring proper since the colors q and r were not incident before the renaming, and clearly no rainbow cycle has been introduced.

Case 1: $M \in \{a, b\}$

Without loss of generality, suppose that $M = b$. Recall we are starting from the assumption that $A \cap B = \emptyset$. We shall show that by valid renaming we can achieve $A \subseteq B$, whence $|A \cup B| = |B| = b = M$.

Since $b = M, b \geq d_G(v) + d_H(v)$, so $|B \setminus D| = b - d_H(v) \geq d_G(v) = |C|$. Therefore we can rename $|C|$ of the colors in $B \setminus D$ with the names of colors in C. This changes B so that now $C \subseteq B \setminus D$, but $(A \setminus C) \cap B = \emptyset$ still.

If $|A \setminus C| = a - d_G(v) \leq d_H(v) = |D|$, then we can rename all of the colors in $A \setminus C$ with the names of colors from D, thus achieving $A \subseteq B$. So assume $a - d_G(v) > d_H(v)$. Let $d_H(v)$ colors in $A \setminus C$ be renamed with the colors in D. This leaves $a - (d_G(v) + d_H(v))$ colors in $A \setminus C$ which are not in B. But we now have $b - (d_G(v) + d_H(v))$ of the original colors in $B \setminus D$ still not renamed (as they shall remain); since $a \leq b = M, a - (d_g(v) + d_H(v)) \leq b - (d_G(v) + d_H(v))$ so we can now rename the $a - (d_G(v) + d_H(v))$ colors in $A \setminus C$ that need renaming with the names of colors in $B \setminus D$ which previously did not appear in A. Thus we achieve $A \subseteq B$ while preserving the PRCF coloring of $G \cup H$.

Case 2: $M = d_G(v) + d_H(v) > \max[a, b]$

Since $a < d_G(v) + d_H(v)$ we have $|A \setminus C| = a - d_G(v) < d_H(v) = |D|$. Thus we can rename all of $A \setminus C$ so that $A \setminus C \subseteq D$. Similarly, we can rename $B \setminus D$ so that $B \setminus D \subseteq C$. With these changes, $|A \cup B| = |C \cup D| = d_G(v) + d_H(v) = M$.

We remind the skeptical reader that in both cases, the color renamings can be performed one color at a time, so that every integer between M and $a + b$ is in $SPRCF(G \cup H)$. □

Corollary 4.4.2. *Let G, H and v be as in Theorem 4.4.1. Suppose that G and H are PRCF-good, and let $a_1 = \min SPRCF(G), a_2 = \min SPRCF(H), b_1 = \max SPRCF(G), b_2 = \max SPRCF(H)$. Then the smallest and largest elements, respectively, of $SPRCF(G \cup H)$ are $\max\{a_1, a_2, d_G(v) + d_H(v)\}$ and $b_1 + b_2$.*

Proof. $M = \max\{a_1, a_2, d_G(v) + d_H(v)\}$ and $b_1 + b_2$ are elements of $SPRCF(G \cup H)$ by Theorem 4.4.1. Suppose that $G \cup H$ is PRCF-colored with exactly x colors appearing. At least a_1 colors must appear on G, and at least a_2 on H; also, $x \geq \chi'(G \cup H) \geq d_G(v) + d_H(v) = d_{G\cup H}(v)$. Thus $x \geq M$.

On the other hand, no more than b_1 colors can appear on G and no more than b_2 on H, so $x \leq b_1 + b_2$. □

Corollary 4.4.3. *Let G, H and v be as in Theorem 4.4.1. If both G and H have proper JL colorings, then so does $G \cup H$.*

Proof. With reference to the proof of Corollary 4.4.2, we are assuming that $b_1 = |V(G)|-1$, $b_2 = |V(H)|-1$, whence $b_1+b_2 = |V(G)|+|V(H)|-2 = |V(G \cup H)|-1$. □

Corollary 4.4.4. *Let G, H and v be as in Theorem 4.4.1. Suppose that G and H are PRCF-solid; let $SPRCF(G) = [a_1, b_1]$ and $SPRCF(H) = [a_2, b_2]$. Then $G \cup H$ is PRCF-solid; in particular, $SPRCF(G \cup H) = [\max\{a_1, a_2, d_G(v) + d_H(v)\}, b_1 + b_2]$.*

Proof. Let $M = \max\{a_1, a_2, d_G(v) + d_H(v)\}$. In view of Corollary 4.4.2 , it suffices to show that if x is an integer and $M < x < b_1 + b_2$, then $x \in SPRCF(G \cup H)$.

It is easy to see (details left to the reader) that as (y_1, y_2) varies over $[a_1, b_1] \times [a_2, b_2]$ the intervals $J(y_1, y_2) = [\max\{y_1, y_2, d_G(v) + d_H(v)\}, y_1+y_2]$ cover $[M, b_1+b_2]$. (Helpful observation: when either y_i is incremented, the right hand endpoint of $J(y_1, y_2)$ is incremented and the left hand endpoint either stays put or is incremented.) The conclusion now follows from Theorem 4.4.1. □

Corollary 4.4.5. *Let G, H and v be as in Theorem 4.4.1. If both G and H are PRCF-excellent, then so is $G \cup H$.*

Proof. This follows from Corollary 4.4.4 and the observations that $|V(G \cup H)| - 1 = (|V(G)| - 1) + (|V(H)| - 1)$ and $\chi'(G \cup H) = \max\{\chi'(G), \chi'(H), d_G(v) + d_H(v)\}$ (left as an exercise). □

As promised earlier, this gives us a second way to prove Theorem 4.1.1.

Proof. Clearly K_1 and K_2 are PRCF-excellent. Every tree of order $n > 2$ can be obtained from a tree of order $n - 1$ by attaching a leaf by a pendant edge incident to some vertex of the smaller tree. Now the result follows by induction on n by applications of Corollary 4.4.5. □

4.5 PRCF spectra of disjoint unions

Conceptually, assuming we can properly edge color each graph H_i with some color palette in a way that avoids rainbow cycles, Lemma 4.3.1 says that we can then carefully choose which colors go into each palette in order to get the following results.

Informally, if G is the disjoint union of graphs H_1 through H_t and we say a_i is in the spectrum of H_i for $1 \leq i \leq t$, then the interval from the largest a_i to the sum of all a_i's is in the spectrum of G. Moreover the spectrum of G is exactly the union of these intervals over all possible choices for each a_i.

Theorem 4.5.1. *Suppose that $G = H_1 + \cdots + H_t$ and $c_i \in SPRCF(H_i), i = 1, \ldots, t$. Then*

$$s(c_1, \ldots, c_t) \subseteq SPRCF(G) = \bigcup_{\substack{a_i \in SPRCF(H_i)\\ i=1,\ldots,t}} s(a_1, \ldots, a_t).$$

Proof. Let each H_i be PRCF-colored with exactly c_i colors appearing. Let A_i be the set of colors appearing on H_i. Then G is PRCF-colored with $|C| = |A_i \cup \cdots \cup A_t|$ colors appearing. By Lemma 4.3.1 we can arrange for $|C|$ to be any integer in $s(c_1, \ldots, c_n)$. This proves the first inclusion, which implies

$$\bigcup_{\substack{a_i \in SPRCF(H_i))\\ i=1,\ldots,t}} s(a_1, \ldots, a_t) \subseteq SPRCF(G).$$

The reverse inclusion follows from the observation that for every PRCF-coloring of G, the restriction of the coloring to H_i is a PRCF-coloring of $H_i, i = 1, \ldots, t$. □

The following theorem give us $SPRCF(G)$ when $G = H_1 + \cdots + H_t$, given that we know the $SPRCF(H_i)$ for each $i = 1, \ldots, t$.

Theorem 4.5.2. *If $H_1, \ldots, H_t$ are PRCF-good, $a_i = \min SPRCF(H_i)$, $b_i = \max SPRCF(H_i)$, $i = 1, \ldots, t$, and $G = H_1 + \cdots + H_t$, then $SPRCF(G) \subseteq [\max\limits_{1 \le i \le t} a_i, \sum_{j=1}^t b_j]$, with equality if each H_i is PRCF-solid.*

Proof. Suppose that $k \in SPRCF(G)$, and suppose that G is PRCF colored so that exactly k colors appear on G. The restriction of this coloring to each H_i is a PRCF coloring of H_i. Therefore if the number of colors appearing on H_i is c_i, we have that $c_i \in SPRCF(H_i)$, so $a_i \le c_i \le b_i$. We have $k \in s(c_i, \ldots, c_t) = [\max\limits_{1 \le i \le t} c_i, \sum\limits_{i=1}^t c_i] \subseteq [\max\limits_{1 \le i \le t} a_i, \sum\limits_{i=1}^t b_i]$. Since k was arbitrary, the inclusion is proved.

If each H_i is PRCF-solid then $SPRCF(H_i) = [a_i, b_i], i = 1, \ldots, t$. The second equality below follows from Theorem 4.5.1. The first equality follows from the following simple observations.

For the reverse containment, with $a_i \le c_i \le b_i$, $i = 1, \ldots, t$, we have $\max\limits_{1 \le i \le t} a_i \le \max\limits_{1 \le i \le t} c_i$ and $\sum_{j=1}^t c_j \le \sum_{j=1}^t b_j$, so $s(c_1, \ldots, c_t) \subseteq [\max\limits_{1 \le i \le t} a_i, \sum_{j=1}^t b_j]$.

For the forward containment, we note that $[\max\limits_{1 \le i \le t} a_i, \sum_{i=1}^t a_i] = s(a_1, \ldots, a_t)$ and then $\left(\sum_{i=1}^t a_i\right) + 1 \in s(a_1 + 1, a_2, \ldots, a_t)$. Continuing this process, iterating each term one at a time, until $s(b_1, \ldots, b_t)$, we see that each $k \in [\max\limits_{1 \le i \le t} a_i, \sum\limits_{j=1}^t b_j]$ is realized. So we have

$$[\max_{1 \le i \le t} a_i, \sum_{j=1}^t b_j] = \bigcup_{(c_1, \ldots, c_t) \in \Pi_{i=1}^t [a_i, b_i]} s(c_1, \ldots, c_t) = SPRCF(G).$$

□

Corollary 4.5.3. *If each component of a graph G is PRCF-excellent then so is G.*

Proof. Suppose that $H_1, \ldots, H_t$ are the components of G, and that each H_i is PRCF-excellent. Then each H_i is PRCF-solid, with $SPRCF(H_i) = [\chi'(H_i), |V(H_i)| - 1]$. The conclusion now follows from Theorem 4.5.2 and the observations that $\chi'(G) = \max_{1 \le i \le t} \chi'(H_i)$ and

$$|V(G)| - (\text{number of components of } G) = |V(G)| - t = \sum_{j=1}^{t} (|V(H_i)| - 1).$$

□

We do not know whether the converse of Corollary 4.5.3 holds, i.e., does the PRCF-excellence of G imply the PRCF-excellence of each component?

4.6 PRCF spectra of graphs formed by joining two PRCF-good graphs by a matching

A **matching** is a non-empty independent set of edges where no two edges in the set share a vertex. Suppose that X and Y are PRCF-good graphs on disjoint vertex sets, and M is a matching, each edge of which has one end in X and the other in Y. Throughout this subsection, $G = X \cup Y \cup M$.

Theorem 4.6.1. *If $a \in SPRCF(X)$ and $b \in SPRCF(Y)$ then $1 + s(a, b) \subseteq SPRCF(G)$.*

Proof. By Theorem 4.5.1 we have $s(a, b) \subseteq SPRCF(X + Y) = SPRCF(X \cup Y)$. Suppose $c \in s(a, b)$. Let $X + Y$ be properly edge colored with exactly c colors appearing while forbidding rainbow cycles. Then the coloring of G obtained by coloring all of the edges of M with a color not appearing among the c colors now has $1 + c$ colors appearing, is proper, and forbids rainbow cycles. (See the argument late in the proof of Proposition 2.2.) □

Corollary 4.6.2. *If X and Y are both PRCF-excellent then* $[\max\{\chi'(X), \chi'(Y)\} + 1, |V(G)| - 1] \subseteq SPRCF(G)$. *Therefore, if* $\chi'(G) = \max\{\chi'(X), \chi'(Y)\} + 1$, *$G$ is PRCF-excellent when both X and Y are.*

Proof. Note that $\max\{\chi'(X), \chi'(Y)\} \le \chi'(G) \le \max\{\chi'(X), \chi'(Y)\} + 1$. From the assumptions that $SPRCF(X) = [\chi'(X), |V(X)| - 1]$ and $SPRCF(Y) =$ (since both are PRCF-excellent), by Theorem 4.6.1 and by the same argument as the one at the end of Theorem 4.5.2, we have

$$\begin{aligned} \bigcup_{\substack{a \in SPRCF(X) \\ b \in SPRCF(Y)}} (s(a, b) + 1) &= [\max\{\chi'(X), \chi'(Y)\} + 1, |V(G)| - 1] \\ &\subseteq SPRCF(G) \\ &\subseteq [\chi'(G), |V(G)| - 1]. \end{aligned}$$

□

Let Q_n denote the n-cube, $n \geq 1$, the graph with vertex set $\{0,1\}^n$, with two binary words of length n adjacent if and only if they differ at exactly one place. Let $\Box$ stand for the cartesian product operation on pairs of graphs. Some facts about Q_n:

1. $|V(Q_n)| = 2^n$.

2. Q_n is regular of degree n.

3. Q_n is bipartite, so $\chi'(Q_n) = \Delta(Q_n) = n$.

4. For $n > 1, Q_n \cong Q_{n-1} \Box K_2$. This means that Q_n can be regarded as 2 disjoint copies of Q_{n-1} joined by a perfect matching.

Theorem 4.6.3. *For all $n > 0, Q_n$ is PRCF-excellent.*

Proof. The proof will be by induction on n. When $n = 1, Q_1 = K_2$ is PRCF-excellent. Suppose that $n > 1$. Since $Q_n = Q_{n-1} \Box K_2$ and $\chi'(Q_n) = \chi'(Q_{n-1}) + 1$, and since Q_{n-1} is PRCF-excellent by our induction hypothesis, the conclusion that Q_n is PRCF-excellent follows from Corollary 4.6.2. $\Box$

5 Critically PRCF-bad graphs

Up to here, the only critically PRCF-bad graphs presented in this paper are K_3 and $K_{2,4}$. The next smallest critically PRCF-bad graph is depicted in Figure 3.

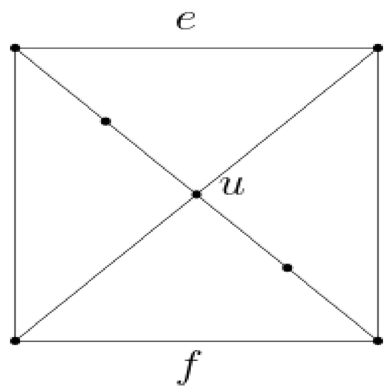


Figure 3: A critically PRCF-bad graph on seven vertices, with ten edges.

The hard part of showing that the graph G in Figure 3 is critically PRCF-bad is in showing that it is PRCF-bad. We will not carry this out here, but here is an observation that might speed up the process for those readers who want to decide for themselves:

If the graph G in Figure 3 does have a PRCF edge coloring, then it has one in which the edges e and f bear the same color, and, of course, the $K_{1,4}$ centered at u is rainbow. The former claim arises from the more general observation that in any PRCF edge coloring of C_4, two opposite (i.e., independent) edges must bear the same color. Since G has 7 different C_4 subgraphs, it is useful to keep this general observation in mind.

We are confident, although not absolutely certain, that there are no critically PRCF-bad graphs other than K_3 and $K_{2,4}$, with G of order less than 7, or of order 7 but with no more than 10 edges, in part because of our own scrutiny as well as because of a computer-generated list provided by co-author G.P. that finds bipartite critically PRCF-bad graphs. All 13 of the graphs in G.P.'s list, and also the graph in Figure 6, are listed in the Appendix of this paper, with the best planar drawings we can manage.

With this list in mind, we find it unlikely that one can find a concise characterization of critical PRCF-badness. However, there is one feature they have in common: they all have a lot of C_4 subgraphs. This observation raises questions: Does there exist a PRCF-bad graph with girth greater than 4? If so, do there exist PRCF-bad graphs with arbitrarily large girth?

If the answer to the second question is yes, then there are infinitely many critically PRCF-bad graphs. The remainder of this section will be devoted to two constructive attempts to prove this theorem directly, due to our co-authors N.T. and M.N. The following easy result will eventually be of assistance.

Proposition 5.1. *If G is vertex-critically-PRCF-bad, then every critically PRCF-bad subgraph of G is spanning in G.*

Proof. If a subgraph H of G is not spanning, then it is a subgraph of $G - v$ for some $v \in V(G)$, and is therefore PRCF-good. □

Since every PRCF-bad graph has a critically PRCF-bad subgraph, we have the following.

Corollary 5.2. *There are infinitely many critically PRCF-bad graphs if and only if there are infinitely many vertex-critically PRCF-bad graphs.*

Proof. Every critically PRCF-bad graph is vertex-critically PRCF-bad, so the existence of infinitely many of the former implies infinitely many of the latter.

For the reverse implication: If G is vertex-critically PRCF-bad, then by Proposition 3.3 G has a critically PRCF-bad subgraph. By Proposition 5.1, every such subgraph has the same order.

The existence of infinitely many vertex-critically PRCF-bad graphs implies the existence of such graphs of infinitely many different orders, which then implies the existence of critically PRCF-bad graphs of infinitely many different orders, which clearly implies the existence of infinitely many critically PRCF-bad subgraphs. □

5.1 Creating PRCF-bad graphs by replacing edges by $K_{2,3}$

The construction is based on Theorem 4.1.3, and especially the fact that there is essentially only one PRCF-coloring of $K_{2,3}$, up to the names of the colors and automorphisms of the graph. The idea is to take a connected graph H, and to replace some or all of its edges by $K_{2,3}$'s as in Figure 4, which also indicates the essentially unique PRCF edge coloring of $K_{2,3}$. (All credit for this subsection goes to our co-author N. Terry.)

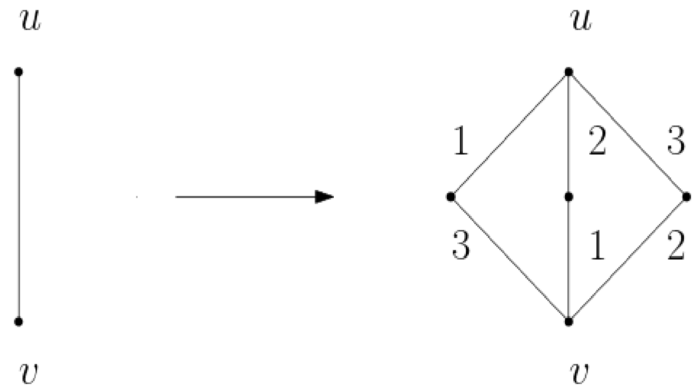


Figure 4: The Terry replacement.

For example, let us look at the graph G obtained from $H = K_3$ by performing the Terry replacement on two of H's three edges.

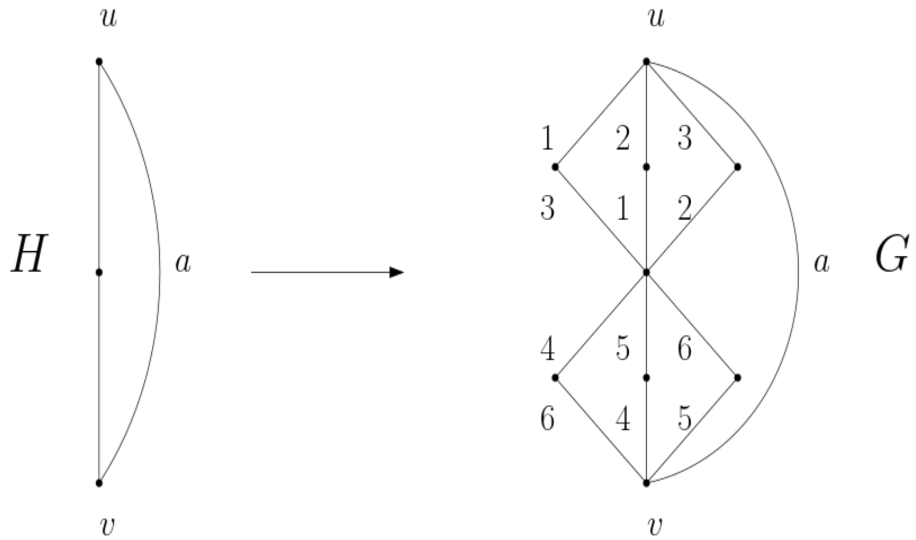


Figure 5: Applying the Terry replacement to two of the three edges of K_3.

In Figure 5 we have given an edge coloring, proper if $a \notin \{1, 2, 3, 4, 5, 6\}$, which is essentially the only possible proper edge coloring of G in which rainbow cycles within the $K_{2,3}$'s are forbidden. If a is chosen so that the coloring is proper, then there are 9 different rainbow cycles in G with this coloring. Thus G is PRCF-bad.

But G is not critically PRCF-bad. In Figure 6 we show a critically PRCF-bad subgraph of G. We leave as an exercise for the reader the verification that the graph in Figure 6 is critically PRCF-bad.

We have not gone far in discovering new critically PRCF-bad graphs using the tool of Terry replacement. The following theorem points to a way, but the way is hard.

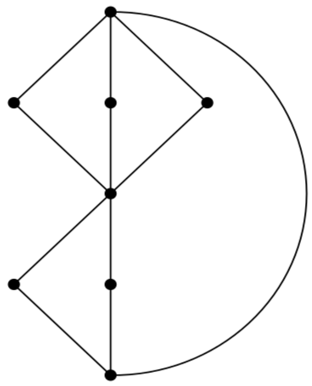


Figure 6: A critically PRCF-bad subgraph of the graph in Figure 5.

Theorem 5.1.1. *Suppose that H is a PRCF-good graph and G is obtained by applying $K_{2,3}$ replacement to all of the edges of H. Then G is PRCF-good.*

Proof. For $e \in E(H)$ let $T(e)$ denote the $K_{2,3}$ replacing e in the formation of G. In any PRCF coloring of $E(G)$, should one exist, the edges of each $T(e)$ will have to bear three colors.

There are at most two kinds of cycles in G: C_4 subgraphs of some $T(e)$, $e \in E(H)$, and cycles not so contained. Each cycle of the latter type is obtainable from a cycle in H by Terry replacement of some of its edges.

Since H is PRCF-good, there is a PRCF coloring of $E(H)$, say with colors $c_1, \dots, c_k$. Colors can be any objects whatever, so we can let $c_1, \dots, c_k$ be pairwise disjoint 3-sets, say $c_j = \{x_j, y_j, z_j\}$. If $e \in E(H)$ is colored with c_j, give $T(e)$ a PRCF coloring with x_j, y_j, z_j. Because the c_j are pairwise disjoint and the coloring of $E(H)$ is proper, the derived coloring of $E(G)$ is proper. We claim that it forbids rainbow cycles in G; if this claim is true then the claim of the theorem is proven.

We chose the coloring so that no C_4 in any $T(e)$, $e \in E(H)$, is rainbow. Suppose that C is a cycle in G not contained in any $T(e)$; let C' be the cycle in H from which C arises by Terry replacement. Because the edge-coloring of H is rainbow-cycle-forbidding, two different edges $e, f \in E(C')$ bear the same color, c_j. Because of the way the edges of $T(e)$ and $T(f)$ are colored with x_j, y_j, z_j, as C traverses each of these two $K_{2,3}$'s, two of these three colors appear on edges of C. Therefore, at least one of x_j, y_j, z_j appears twice on the edges of C. □

The graph G in Figure 7, touted there as the only critically PRCF-bad subgraph of the graph obtained from K_3 by Terry replacement of every edge, is also the result of Terry-replacing two adjacent edges of C_4. This shows that if the statement of Theorem 5.1.1 is altered by replacing "all" with "some", the resulting statement is not a theorem.

Moreover, Theorem 5.1.1 does not say that a graph obtained from a PRCF-bad graph by total Terry replacement (i.e., by Terry replacement of every edge) will be PRCF-bad, but it does say that if one is searching for new PRCF-bad graphs by total Terry replacement, then do not bother starting with PRCF-good graphs. In fact, a little thought shows that one may as well apply total Terry replacement to critically PRCF-bad graphs in search of PRCF-bad graphs from which to extract new critically PRCF-bad graphs.

If G is obtained from H by total Terry replacement, we write $G = T(H)$.

The following two lemmas posit for reference some obvious facts about PRCF-goodness and PRCF-criticality. Proofs are omitted.

Lemma 5.1.2. *(a) If G is vertex-critically-PRCF-bad, then G is connected and has no vertices of degree 1 (leaves).*

(b) If a graph G is critically PRCF-bad, then G is vertex-critically-PRCF-bad.

Lemma 5.1.3. *If G is PRCF-good, then a graph obtained from G by adding a leaf or an isolated vertex is also PRCF-good.*

Proposition 5.1.4. *If $G = T(H)$, then G is critically PRCF-bad if and only if G is vertex-critically-PRCF-bad.*

Proof. The "only if" claim holds by Lemma 5.1.2 part (b). Suppose that G is vertex-critically-PRCF-bad. Because $G = T(H)$, every $e = uv \in E(G)$ is an edge of some $K_{2,3}$ subgraph of G in which one of u, v (say v) has degree 2 in G. Therefore, $G - e$ is obtained from the PRCF-good graph $G - v$ by adding a leaf (namely, v). By Lemma 5.1.3, $G - e$ is PRCF-good. Since $e \in E(G)$ was arbitrary, it follows that G is critically PRCF-bad. □

Proposition 5.1.4 has not helped, so far, in our hunt for critically PRCF-bad graphs. Our plan was to apply Proposition 5.1.4 with H being a critically PRCF-bad graph. Success would involve showing that $G = T(H)$ is PRCF-bad and then that $G - u$ is PRCF-good for every $u \in V(G)$.

For all of the small critically PRCF-bad graphs H that we have checked, $T(H)$ is indeed PRCF-bad, but not critically so. What is more, verifying that $T(H)$ is PRCF-bad, even for these small critically PRCF-bad graphs, is not easy. We strongly suspect that $T(H)$ will be PRCF-bad (though not critically so) whenever H is PRCF-bad, but we do not yet have a proof.

We are interested in extracting critically PRCF-bad graphs from $T(H)$, when H is a critically PRCF-bad graph, because it may lead to a proof that there are infinitely many critically PRCF-bad graphs. We will finish this section with the result of this process in the simplest case, when $H = K_3$.

Lemma 5.1.5. *Suppose that H is vertex-critically-PRCF-bad, G is a subgraph of $T(H)$ and $v \in V(H) \cap V(G)$. Then $G - v$ is PRCF-good.*

Proof. $G - v$ is a subgraph of $T(H - v)$ plus, possibly, some added leaves. Since $H - v$ is PRCF-good, it follows that $T(H - v)$ is PRCF-good, by Theorem 5.1.1. Therefore every subgraph of $T(H - v)$ is PRCF-good, and so G is PRCF-good, by Lemma 5.1.3. □

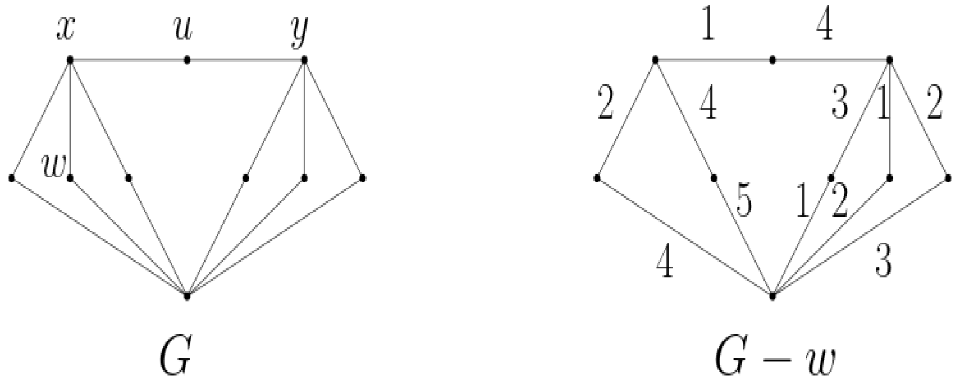


Figure 7: G is the unique (up to isomorphism) critically PRCF-bad subgraph of $T(K_3)$. A PRCF coloring of $G - w$ is given.

Proposition 5.1.6. *The first graph G in Figure 7 is the only (up to isomorphism) critically PRCF-bad subgraph of $T(K_3)$.*

Proof. Cleary G is a subgraph of $T(K_3)$. Suppose G has a PRCF coloring. Without loss of generality, the left $K_{2,3}$ has the unique PRCF coloring of $K_{2,3}$ with colors 1, 2, and 3, and the right $K_{2,3}$ is so-colored with colors 4, 5, and 6. Then the color of uy cannot be one of 4, 5, or 6, and the color of xu cannot be one of 1, 2, or 3. Consequently, there must be a rainbow C_6 in G, with this coloring; so no such coloring exists. Thus G is PRCF-bad.

Since removing any edge of G leaves a leaf, to show that G is critically PRCF-bad it suffices to show that removing any vertex of G of degree 2 leaves a PRCF-good graph. It is obvious that $G - u$ is PRCF-good. Without loss of generality, it now suffices to show that $G - w$ has a PRCF-coloring. This is displayed in Figure 7.

It remains to show that G is, up to isomorphism, the *only* critically PRCF-bad subgraph of $T(K_3)$. By Lemma 5.1.3 and Lemma 5.1.5 (with $H = K_3$), to find critically PRCF-bad subgraphs of $T(K_3)$ it suffices to consider graphs obtained from $T(K_3)$ by removing vertices of degree 2; for instance, the graph G in Figure 7 is obtained by removing from $T(K_3)$ two vertices of degree 2 that inhabit the same $K_{2,3}$ in $T(K_3)$. Next in line, in both the ordering by order and the subgraph ordering, is the graph in Figure 8, for which a PRCF coloring is supplied. □

The graph in Figure 6 is not a subgraph of $T(K_3)$, but of the graph obtained by Terry-replacing 2 of K_3's 3 edges. It is the unique critically PRCF-bad subgraph of that K_3 derivative.

The graph obtained by Terry-replacing only one of the edges of $K_{2,3}$ is $K_{2,4}$. It can be shown that $T(K_{2,4})$ has a critically PRCF-bad proper subgraph, and is therefore PRCF-bad, but not critically so; but we shall not pursue this matter further.

5.2 Mycielskians of odd cycles

The Mycielskian, $M(G)$, of a finite simple graph G on n vertices is formed as follows

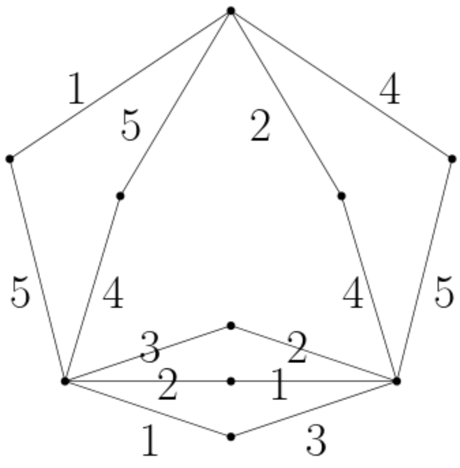


Figure 8: A PRCF coloring of another subgraph of $T(K_3)$.

1. Put a copy of G on vertices $x_1, \ldots, x_n$.
2. For each x_i introduce its "clone" y_i and make y_i adjacent in $M(G)$ to each x_j for which $x_i x_j \in E(G)$. There are no edges $y_i y_j$ in $M(G)$.
3. Finally, introduce a vertex z which is adjacent to each y_i, and to none of $x_1, \ldots x_n$ in $M(G)$.

$M(C_5)$, the famous Grötzsch graph, is depicted in Figure 9.

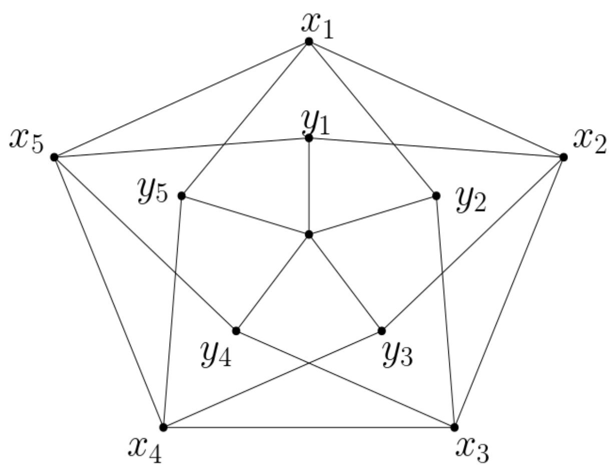


Figure 9: The Grötzsch graph, $M(C_5)$

Theorem 5.2.1. *If $n \geq 5$ is an odd integer, then $M(C_n)$ is PRCF-bad.*

Proof. Let $x_1, \ldots, x_n$ be the vertices of C_n with x_i adjacent to x_j when $|i - j| = 1$ or when $i = 1$ and $j = n$. Let $y_1, \ldots, y_n$ and z be as described in the instructions for the formation of $M(C_5)$. See Figure 10.

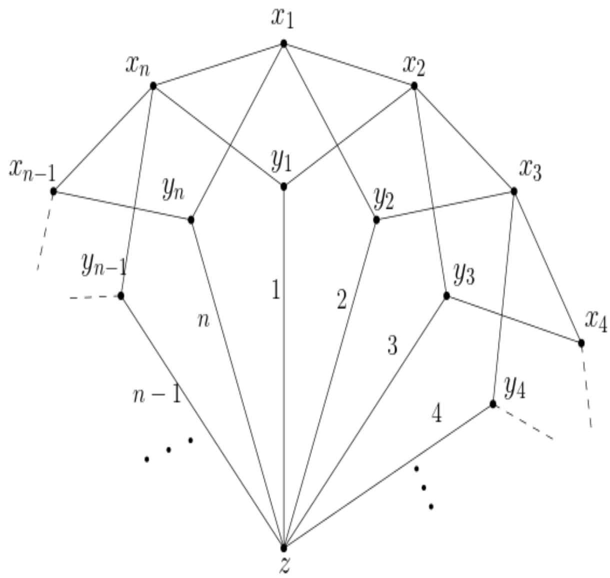


Figure 10: Part of $M(C_n)$, with zy_i colored i, for $1 \leq i \leq n$.

Suppose that $\phi : E(M(C_n)) \to \{1, 2, \dots\}$ is a PRCF coloring of $M(C_n)$. Since ϕ is a proper coloring, we may as well suppose that $\phi(zy_i) = i$, for $1 \leq i \leq n$.

Reading indices modulo n, observe that z, y_k, x_{k+1}, y_{k+2} induce a C_4 in $M(C_n)$ for each $k = 1, \dots, n$, with edges zy_k and zy_{k+2} colored k and $k + 2$, respectively. Therefore, either $y_k x_{k+1}$ is colored $k + 2$ or $x_{k+1}y_{k+2}$ is colored k. We can form a binary word $\bar{a} = a_1 \dots a_n \in \{L, R\}^n$ such that $a_k = R$ implies that $\phi(x_{k+1}y_{k+2}) = k$ and $a_k = L$ implies that $\phi(y_k x_{k+1}) = k + 2$. Imagine that the word is arranged in a circle – that is, this word represents a coloring of the vertices of C_n with the colors L and R.

Either $\bar{a} = L^n$, or $\bar{a} = R^n$, or both R and L appear in $\bar{a}$. Because n is odd, in the latter case either the arc RRL of consecutive colors appears in $\bar{a}$, or the arc RLL appears in $\bar{a}$. Without loss of generality, assume that $a_1a_2a_3 = RRL$.

Referring to Figure 11, observe that, under our assumption, the P_4's $x_2y_3zy_4x_3$ and $x_3y_4zy_3x_4$ are both rainbow. Because ϕ is a PRCF coloring, it follows that $\{\phi(x_2x_3), \phi(x_3x_4)\} = \{3, 4\}$; but then the 4-cycle $x_2y_3x_4x_3x_2$ is rainbow.

Therefore, we may assume, without loss of generality, that $\bar{a} = R^n$.

Referring to Figure 12, observe that for each $k = 1, \dots, n$, the P_4 $x_k y_{k+1} z y_{k+2} x_{k+1}$ is rainbow. Since the 5-cycle induced by these vertices cannot be rainbow, it follows that $\phi(x_k x_{k+1}) \in \{k + 1, k + 2\}$, for $k = 1, \dots, n$. (We are again reading indices mod n.)

If, for instance, $\phi(x_1x_2) = 3$, then the properness of ϕ forces $\phi(x_2x_3)$=4, which forces $\phi(x_3x_4) = 5$, etc., and we have a rainbow cycle, C_n. Similarly, if $\phi(x_1x_2) = 2$, the forcing proceeds counterclockwise, and again we have that C_n is rainbow. Thus there is no PRCF coloring of $M(C_n)$, so $M(C_n)$ is PRCF-bad. □

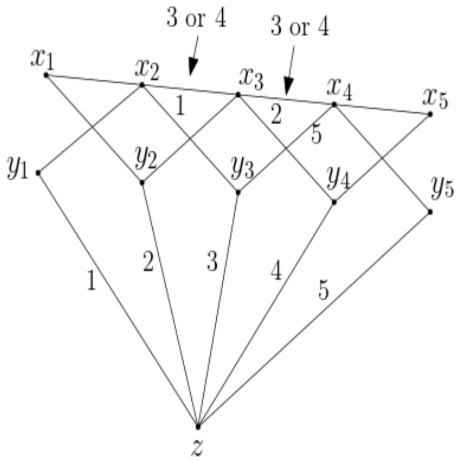


Figure 11: The coloring ϕ partially revealed.

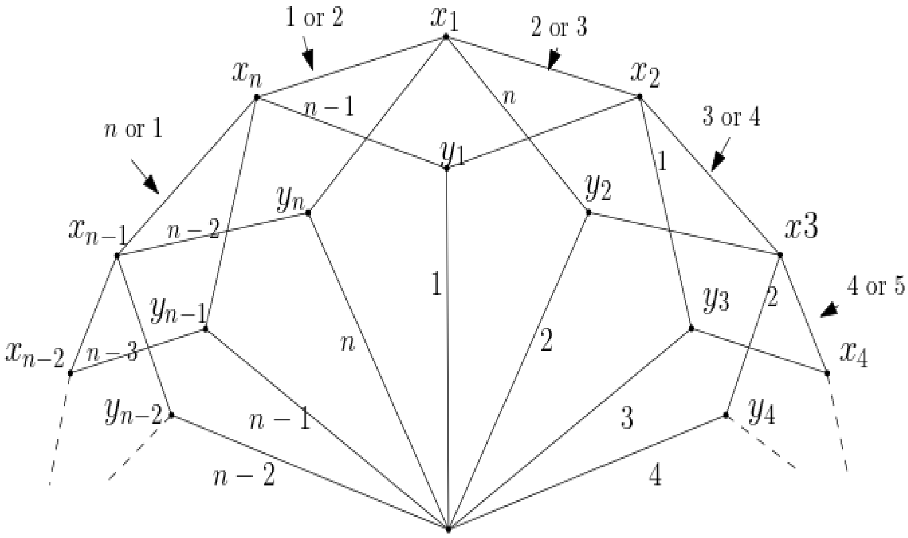


Figure 12: ϕ partially revealed, under the assumption that $\bar{a} = R^n$.

We strongly suspect that when $N \geq 5$ is odd, then $M(C_n)$ is vertex-critically PRCF-bad, which, if true, would show that there are infinitely many critically PRCF-bad graphs, by Corollary 5.2.

However, we can prove that there are infinitely many vertex-critically-PRCF-bad graphs, and thus infinitely many critically PRCF-bad graphs, without actually exhibiting an infinite sequence of such graphs.

Theorem 5.2.2. *For $n \geq 3$, let the vertices of $M(C_n)$ be $x_1, \dots x_n, y_1, \dots, y_n, z$ as in the proof of Theorem 5.2.1. For each $i \in \{1, \dots, n\}$, $M(C_n) - \{x_i, y_i\}$ is PRCF-good.*

Proof. Clearly, it suffices to prove that $H_{n-1} = M(C_n) - \{x_n, y_n\}$ is PRCF-good. We depict H_m, for $m \geq 2$, in Figure 13, with a coloring $\phi = \phi_m : E(H_m) \to \{1, \dots, m+2\}$ that we claim is a PRCF coloring.

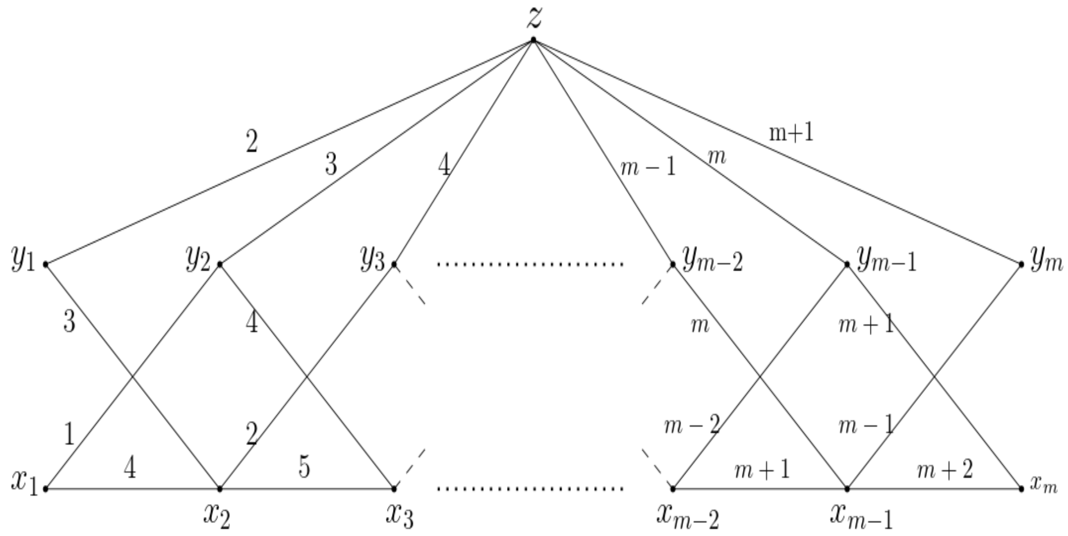


Figure 13: $H_m = M(C_{m+1}) - \{x_{m+1}, y_{m+1}\}$, with a PRCF edge coloring.

$\phi = \phi_m$ is defined by :

$$\begin{gathered}\phi(zy_i) = i+1, \text{ for } i = 1, \dots, m;\\ \phi(x_{i+1}y_i) = i+2, \text{ for } i = 1, \dots, m-1;\\ \phi(x_iy_{i+1}) = i, \text{ for } i = 1, \dots, m-1;\\ \phi(x_ix_{i+1}) = i+3, \text{ for } i = 1, \dots, m-1.\end{gathered}$$

Observe that if $m \geq 3$ then the restriction of ϕ_m to the edges of $H_m - \{x_m, y_m\}$ is ϕ_{m-1}.

Checking the colors on edges incident to each vertex of H_m shows that ϕ_m is a proper edge coloring. We will show that ϕ_m forbids rainbow cycles by induction on m, starting with $m = 2$; H_2 is C_5 and $\phi_2(zy_2) = 3 = \phi(x_2y_1)$.

Now suppose that $m \geq 3$ and that ϕ_{m-1} is a PRCF coloring of H_{m-1}. Then to show that there are no rainbow cycles in H_m with the coloring ϕ_m it suffices to show that neither x_m nor y_m is in such a rainbow cycle.

Any cycle containing both x_m and y_m must contain the edges zy_m and x_my_{m-1}, both colored $m+1$. So now we must consider cycles containing exactly one of x_m and y_m.

Suppose that C is a rainbow cycle in H_m (with reference to ϕ_m) such that $y_m \in V(C)$ and $x_m \notin V(C)$. Then, because y_m is incident to only two edges, those edges zy_m and $x_{m-1}y_m$, colored $m+1$ and $m-1$, respectively, are in $E(C)$. Let us ask: what other edges *must* be in $E(C)$?

Because $x_m \notin V(C)$, $x_{m-1}x_m \notin E(C)$, and because $\phi_m(x_{m-2}x_{m-1}) = m+1$, $x_{m-2}x_{m-1} \notin E(C)$. Therefore, $x_{m-1}y_{m-2}$, colored m, must be in $E(C)$.

Because $\phi(zy_{m-2}) = m-1$, which already appears as a color on C, on $x_{m-1}y_m$, it must be that $x_{m-3}y_{m-2} \in E(C)$, adding $m-3$ to the list of colors appearing on C. Examining the colors on the other three edges incident to x_{m-3}, we find ourselves forced to leave x_{m-3} along $x_{m-3}y_{m-4}$, adding $m-2$ to the list of colors on C.

Continuing in this way, we find that the coloring ϕ_m and the assumptions that C is rainbow, that $y_m \in C$, and that $x_m \notin C$, force the existence of a path P on C starting at z: $z, y_m, x_{m-1}, y_{m-2}, \ldots$ landing on either x_1 or y_1, depending on whether m is even or odd. On the subpath from z to $x_{m-k}, k > 1$, the colors appearing are $\{m+1, m, m-1, \ldots m-k\} \setminus \{m-k+1\}$; on the subpath from z to y_{m-k} the colors appearing are $\{m+1, m, \ldots, m-k+1\}$. From this it can be deduced that after arriving at $x_{m-k}, 1 \leq k \leq m-2$, the next edge on the path can only be $x_{m-k}y_{m-k-1}$, and after arriving at y_{m-k}, $2 \leq k \leq m-2$, the next edge on the path can only be $x_{m-k-1}y_{m-k}$.

If y_1 is a stop on this path then the colors appearing along the path from z to y_1 are $\{2, \ldots m+1\}$; because y_1 was arrived at along x_2y_1, the only edge along which to depart from y_1 to complete the cycle is y_1z, which is colored 2–thus the rainbow C cycle cannot exist. If x_1 is a stop on the path, arrived at along x_1y_2, then the colors from z to x_1 along P are $\{1, \ldots m+1\} \setminus \{2\}$; the only edge to leave x_1 by x_1x_2 colored 4. Again, the rainbow cycle C cannot exist.

Now suppose that C is a rainbow cycle in H_m, with reference to ϕ_m, such that $x_m \in V(C)$ and $y_m \notin C$. The proof that C does not exist starts like the proof just completed, except that we build a path on C containing x_m by building out from x_m in both directions.

Since x_m has degree 2 in H_m, $x_m \in V(C)$ implies that $x_{m-1}x_m, x_my_{m-1} \in E(C)$, bearing colors $m+2, m+1$, respectively. Since $y_m \notin V(C)$ and $x_{m-2}x_{m-1}$ bears color $m+1$, the other edge of C incident to x_{m-1} must be $x_{m-1}y_{m-2}$, adding m to the colors that must appear on C, on the edges so far discovered.

That means that $zy_{m-1} \notin E(C)$, so $x_{m-2}y_{m-1} \in E(C)$, adding $m-2$ to the color list, which now stands at $m+2, m+1, m, m-2$. Our path has end vertices x_{m-2} and y_{m-2} at this point. If $m = 3$ then it is straightforward to see that this path cannot be extended to a rainbow cycle in H_3; so we assume that $m > 3$.

Examining the edges incident to x_{m-2}, we see that it must be that $x_{m-2}y_{m-3} \in E(C)$, adding $m-1$ to the list of colors known to be on C, and forcing $x_{m-3}y_{m-2} \in E(C)$. The list of colors known to be on C is now $m+2, m+1, m, m-1, m-2, m-3$. Since the colors on the 3 edges of H_m besides $x_{m-3}y_{m-2}$ incident to x_{m-3} (if $m > 4$) are among those on this list, there is no such rainbow cycle C satisfying the assumptions of this part of the proof. If $m = 4$, then $x_{m-3} = x_1$, so the same conclusion holds. □

Corollary 5.2.3. *If $n \geq 5$ is odd, then $M(C_n)$ has a vertex-critically-PRCF-bad subgraph of order at least n. In fact, every vertex-critically-PRCF-bad subgraph of $M(C_n)$ has order at least n.*

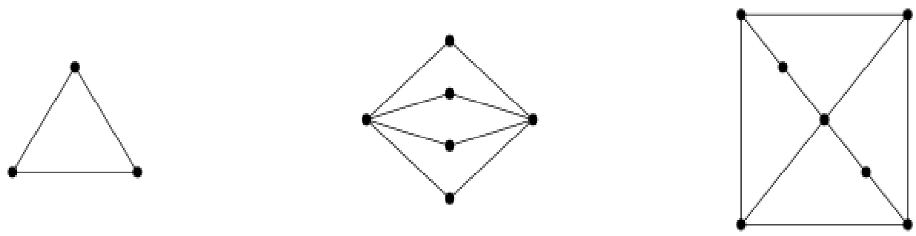
Proof. Every vertex-critically-PRCF-bad subgraph of $M(C_n)$ is obtained by removing vertices from $M(C_n)$. The order of $M(C_n)$ is $2n+1$; any $(n+2)$-set of vertices of $M(C_n)$ must contain a set $\{x_i, y_i\}$ for some $i \in \{1, \ldots, n\}$. Therefore , every subgraph of $M(C_n) - S$, $S \subseteq V(M(C_n))$, $|S| = n+2$, is PRCF-good. Consequently every PRCF-bad subgraph of $M(C_n)$ has order greater than $(2n+1)-(n+2) = n-1$. □

Corollary 5.2.4. *There are infinitely many different critically PRCF-bad graphs.*

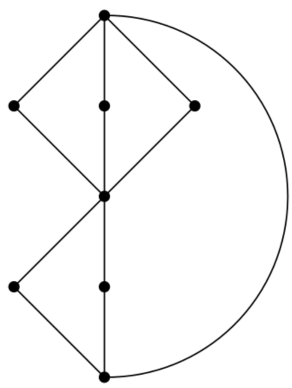
Proof. This follows from Corollaries 5.2 and 5.2.3. □

Appendix: Small critically PRCF-bad graphs

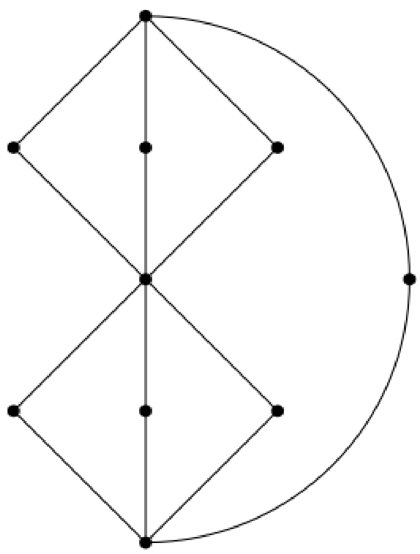
Our fifth co-author (G.P.) wrote a computer program for the task of discovering all bipartite critically PRCF-bad graphs of order at most 12. The program discovered three graphs that we knew of already (A_2, A_3, and A_5, below) and ten others. Of these ten, we have hand-checked the PRCF-bad criticality of none; nor have we confirmed that Greg's program discovered all bipartite critically PRCF-bad graphs of order at most 12. As for non-bipartite critically PRCF-bad graphs, we know of only the two given below, A_1 and A_4. Below is a list of what we believe to be the first 15 critically PRCF-bad graphs, listed by order.



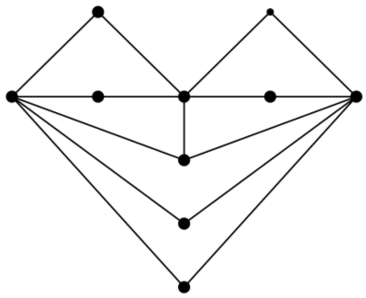
K_3, $K_{2,4}$, and the graph in Figure 3 that we will call A_1, A_2, andA_3 respectively.



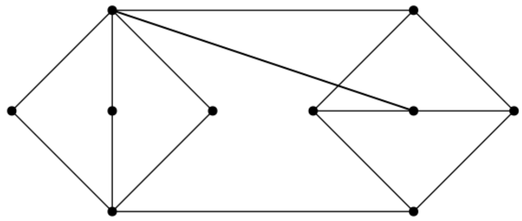
A_4: This appears in Figure 6. It is not bipartite.



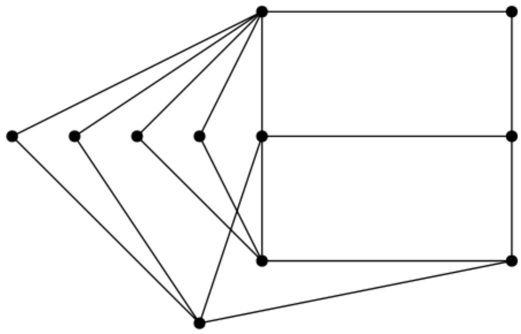
A_5: This bipartite graph is a subgraph of $T(K_3)$. It appears, drawn differently, in Figure 7.



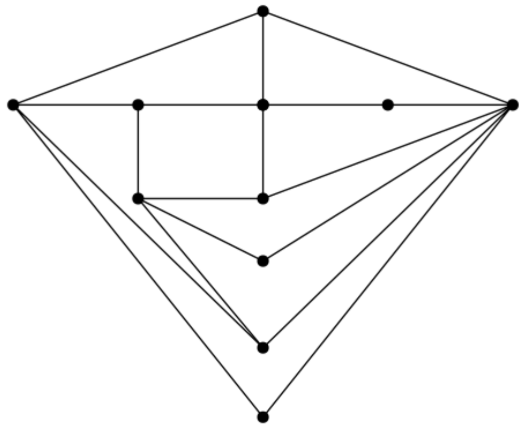
A_6



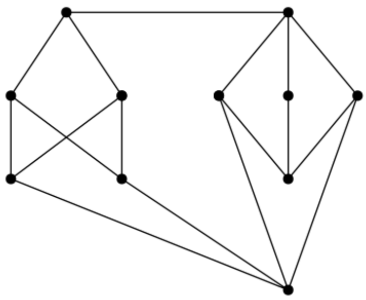
A_7



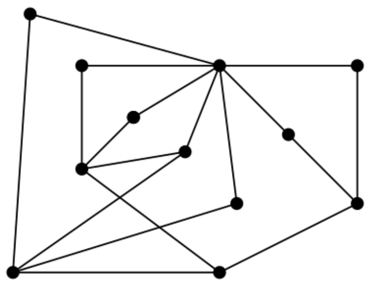
A_8



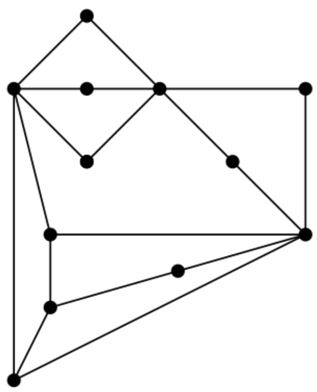
A_9



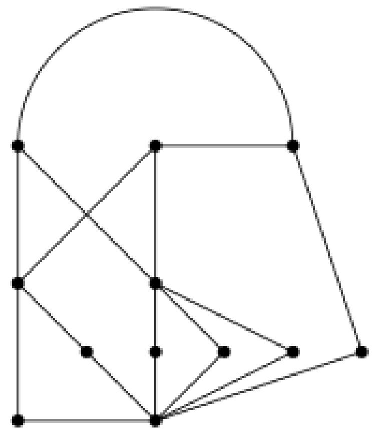
A_{10}



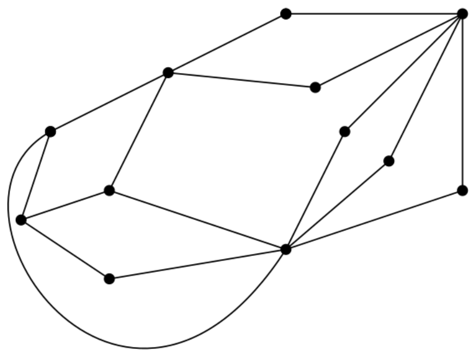
A_{11}



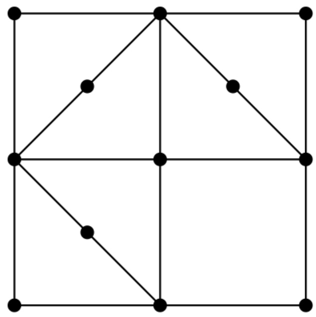
A_{12}



A_{13}



A_{14}



A_{15}

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