AUSTRALASIAN JOURNAL OF COMBINATORICS
Volume **90(3)** (2024), Pages 341–356

Quasisymmetric invariants for families of posets

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Abstract

We address the question whether certain classes of labeled posets can be distinguished by their P-partition enumerator. Recent progress has been made for naturally labeled posets. We investigate the general question and prove that the P-partition enumerator does distinguish classes of labeled posets that we call: fair series-parallel posets; cypress trees; and centipedes.

1 Introduction

1.1 *Distinguishing* combinatorial objects, a classical problem

Consider an invariant Γ on a class of graphs $\mathcal{G}$. It is a basic question, both classical and active in recent years, to decide whether Γ is injective on (isomorphic classes of) $\mathcal{G}$. In this case we say that Γ *distinguishes* elements of $\mathcal{G}$.

Let us start by recalling the definition of the celebrated chromatic polynomial χ. In the following definitions, all graphs are finite and simple.

Definition 1.1. A (proper) *coloration* of a graph $G = (V, E)$ is a function

$$c : V \longrightarrow \mathbb{N}^*$$

such that for $v, v' \in V$, $(v, v') \in E$ implies that $c(v) \neq c(v')$.

It is well-known that the number of colorations of G over t colors turns out to be a polynomial in t, which we denote by $\chi_G(t)$, the *chromatic polynomial* of G. It is true that some information can be derived from χ_G, such as the number of vertices $|V|$, the number of edges $|E|$, or the number of connected components of G. But

ISSN: 2202-3518

χ is very far from being able to distinguish even simple classes of graphs, since for example $\chi_T(t) = t(t-1)^{k-1}$ for any tree T with k vertices.

In order to distinguish classes of graphs, Stanley defined in 1995 a new chromatic invariant, stronger than the chromatic polynomial [19]. Let $\mathbf{x} = x_1, x_2, \ldots$ denote an infinite commutative alphabet.

Definition 1.2. The *chromatic symmetric function* X_G of a graph G is defined as

$$X_G(\mathbf{x}) = \sum_c \mathbf{x}^c$$

where the sum ranges over the colorations c of G, and $\mathbf{x}^c$ stands for $\prod_{i \in \mathbb{N}^*} x_i^{|c^{-1}(i)|}$.

In his original paper, Stanley observed that X_G does not distinguish graphs in general: he gave the example of two graphs on 5 vertices with equal chromatic symmetric functions. But he conjectured the following.

Conjecture 1.3. *The chromatic symmetric function distinguishes trees: for any pair of trees T_1 and T_2, if $X_{T_1} = X_{T_2}$, then T_1 and T_2 are isomorphic.*

This conjecture has been intensively studied and proved for special subclasses of trees [7, 8, 12, 15]. Important variations have been proposed [3, 13, 17], but the conjecture is still open in general.

Let us now turn to an analogue of the previous question, in the setting of (labeled) posets instead of graphs.

1.2 The main question of this work: quasisymmetric invariant distinguishing labeled posets

Let P be a poset with n elements; we write $|P| = n$. Denote the order relation on P by $\leq_P$, to avoid the confusion with the usual order on the positive integers, which we shall denote by $\leq$. A *labeling* of P is a bijection $\omega : P \to [n]$. A *labeled poset* (P, ω) is then a poset P with an associated labeling ω.

Definition 1.4. For a labeled poset (P, ω), a (P, ω)*-partition* is a map f from P to the positive integers satisfying the following two conditions:

- if $a <_P b$, then $f(a) \leq f(b)$, i.e. f is order-preserving;
- if $a <_P b$ and $\omega(a) > \omega(b)$, then $f(a) < f(b)$.

In other words, a (P, ω)-partition is an order-preserving map from P to the positive integers with certain strictness conditions determined by ω. Examples of (P, ω)-partitions f are given in Figure 2.

The meaning of the double edges in the figure follows from the following observation about Definition 1.4. For $a, b \in P$, we say that a is *covered* by b in P, denoted $a \prec_P b$, if $a <_P b$ and there does not exist c in P such that $a <_P c <_P b$. Note that a definition equivalent to Definition 1.4 is obtained by replacing both appearances of

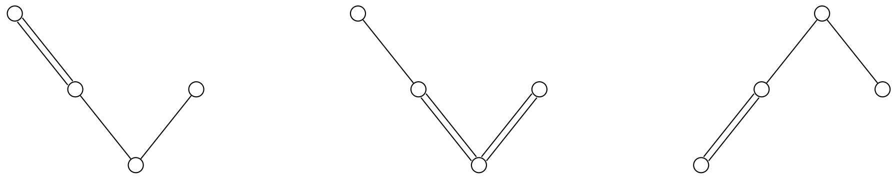


Figure 1: An edge-decorated poset P (left), together with $\overline{P}$ (middle), and $P^{\updownarrow}$ (right).

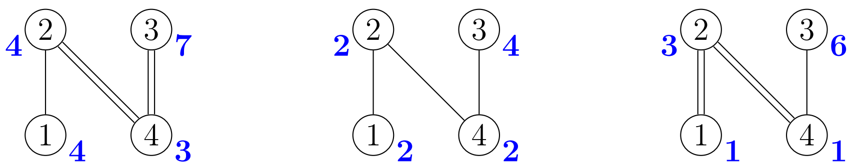
the relation $a <_P b$ with the relation $a \prec_P b$. In other words, we require that f be order-preserving along the edges of the Hasse diagram of P, with $f(a) < f(b)$ when the edge $a \prec_P b$ satisfies $\omega(a) > \omega(b)$. With this in mind, we will consider those edges $a \prec_P b$ with $\omega(a) > \omega(b)$ as *strict edges* and we will represent them in Hasse diagrams by double lines. Similarly, edges $a \prec_P b$ with $\omega(a) < \omega(b)$ will be called *weak edges* and will be represented by single lines.

From the point of view of (P, ω)-partitions, the labeling ω only determines which edges are strict and which are weak. Therefore, we say that two labeled posets (P, ω) and (Q, ω') are *isomorphic* if P and Q are isomorphic as posets and they have equivalent sets of strict and weak edges according to a poset isomorphism.

Definition 1.5. An *edge-decorated poset* is a poset P such that each edge in the Hasse diagram of P is assigned to be either weak or strict.

From now on, we will consider edge-decorated posets P instead of labeled posets (P, ω), and we will use the word *P-partition* instead of (P, ω)-partition.

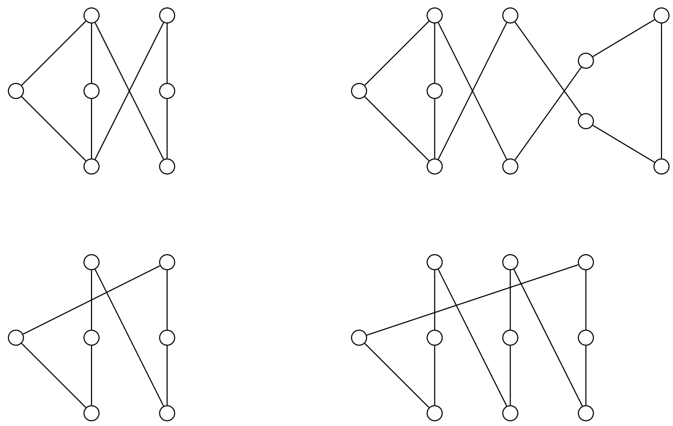
Moreover, we shall use the notation: $\overline{P}$ for the edge-decorated poset obtained by switching weak and strict edges in P, and $P^{\updownarrow}$ for the reverse (upside-down) poset with the same decoration of edges. These notions are illustrated by Figure 1.



(a) (b) (c)

Figure 2: Examples of P-partitions (the images are written in bold and blue next to the nodes)

If all the edges of P are weak, as in Figure 2(b), P is said to be *weak*. This correponds to order-preserving labelings ω, and such P are called *naturally labeled* in some references.



(a) (b)

Figure 3: Pairs of weak posets with equal P-partition enumerators

Definition 1.6. Let P be an edge-decorated poset. The well-known *P-partition enumerator* is defined by

$$K_P(\mathbf{x}) = \sum_f x_1^{\#f^{-1}(1)} x_2^{\#f^{-1}(2)} \cdots \tag{1}$$

where the sum ranges over all P-partitions $f : P \to \mathbb{N}^*$.

A poset being a tree simply means its Hasse diagram is a tree. The following conjecture is presented in [1].

Conjecture 1.7. *The P-partition enumerator distinguishes weak trees.*

If we try to relax the conditions in Conjecture 1.7, we obtain easily false statements, even for weak posets. The first example of non-isomorphic weak posets with the same K was given in [16] and appears in Figure 3(a). A *bowtie* is the poset consisting of elements a_1, a_2, b_1, b_2 with cover relations $a_i < b_j$ for all i, j. Notice that each poset in Figure 3(a) has a bowtie as an induced subposet. Otherwise, we say the poset is *bowtie-free*. Weakening the tree hypothesis of Conjecture 1.7 to bowtie-free results in a false statement, with Figure 3(b) being the smallest counterexample.

An important result in this context is the following.

Theorem 1.8 ([5], Theorem 1.3)**.** *The P-partition enumerator distinguishes weak* rooted *trees.*

But Conjecture 1.7 is still open. Theorem 1.8 was generalised to weak series-parallel posets in [11]. Up to very recently, the effort put on distinguishing posets through the P-partition enumerator has been focused on weak posets (we may add [10, 20] to the references already mentioned). When we consider general edge-decorated posets (not necessarily weak), things are getting (way!) harder. Figure 4 shows how very simple edge-decorated posets may have the same K function. Of

Figure 4: Two simple edge-decorated posets with equal P-partition enumerator

course, this example implies that there is no chance to extend Conjecture 1.7 to general edge-decorated trees.

Up to now, few results were obtained about distinguishing edge-decorated posets by their P-partition enumerator. We may cite [1], where the case of a family of named (rooted) fair trees was treated, and we propose here a generalization of this result (in Section 4).

It is the purpose of this article to present new results on this question. We prove here that the P-partition enumerator does distinguish families named: fair series-parallel posets (Section 4), cypress trees (Section 5), and centipedes (Section 6).

2 Definitions and useful tools

2.1 Quasisymmetric functions

We shall give here basic definitions and properties about quasisymmetric functions.

For our purposes, quasisymmetric functions are elements of $\mathbb{Q}[[x_1, x_2, \ldots]]$ and we denote the ring of quasisymmetric functions by QSym. We will make use of both of the classical bases for QSym. If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ is a composition of n, then we define the *monomial quasisymmetric function* M_α by

$$M_\alpha = \sum_{i_1 < i_2 < \ldots < i_k} x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_k}^{\alpha_k}.$$

We recall that compositions $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$ of n are in bijection with subsets of $[n-1]$, and let $S(\alpha)$ denote the set $\{\alpha_1, \alpha_1 + \alpha_2, \ldots, \alpha_1 + \alpha_2 + \cdots + \alpha_{k-1}\}$. Thus we also denote M_α by $M_{S(\alpha),n}$. Notice that these two notations are distinguished by the latter one including the subscript n; this subscript is helpful since $S(\alpha)$ does not uniquely determine n.

The second classical basis is composed of the *fundamental quasisymmetric functions* F_α defined by

$$F_\alpha = F_{S(\alpha),n} = \sum_{S(\alpha) \subseteq T \subseteq [n-1]} M_{T,n}\,. \tag{2}$$

The relevance of this latter basis to $K_{(P,\omega)}$ is due to Theorem 2.1 below.

Recall that any permutation $\pi \in S_n$ has an associated descent set $\mathrm{des}(\pi)$ given by $\{i \in [n-1] : \pi(i) > \pi(i+1)\}$. We will call the corresponding composition of n the *descent composition* of the permutation π, denoted $\mathrm{co}(\pi)$. As an example, if $\pi = 243561$, then $\mathrm{des}(\pi) = \{2, 5\}$ and $\mathrm{co}(\pi) = 231$. Let $\mathcal{L}(P, \omega)$ denote the set of all linear extensions of P, regarded as permutations of the ω-labels of P. For example, for the labeled poset in Figure 2(a), $\mathcal{L}(P, \omega) = \{1423, 1432, 4123, 4132\}$.

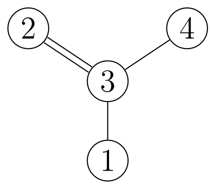


Figure 5: The labeled poset of Example 2.2

Theorem 2.1 ([4, 18])**.** *Let (P,ω) be a labeled poset with $|P| = n$. Then*

$$K_{(P,\omega)} = \sum_{\pi\in\mathcal{L}(P,\omega)} F_{\mathrm{des}(\pi),n} = \sum_{\pi\in\mathcal{L}(P,\omega)} F_{\mathrm{co}(\pi)}\,.$$

Example 2.2. The labeled poset (P,ω) of Figure 5 has $\mathcal{L}(P,\omega) = \{1324, 1342\}$ and hence

$$\begin{aligned} K_{(P,\omega)} &= F_{\{2\},4} + F_{\{3\},4} \\ &= F_{22} + F_{31} \\ &= (M_{\{2\},4} + M_{\{1,2\},4} + M_{\{2,3\},4} + M_{\{1,2,3\},4}) \\ &\quad + (M_{\{3\},4} + M_{\{1,3\},4} + M_{\{2,3\},4} + M_{\{1,2,3\},4}) \\ &= M_{22} + M_{31} + M_{112} + 2M_{211} + M_{121} + 2M_{1111}. \end{aligned}$$

Remark 2.3. It is easy to deduce the P-partition enumerator of the edge-decorated poset $\overline{P}$ obtained by exchanging weak and strict edges in P. We may call $\overline{P}$ the *dual* of P. Indeed, if $K_P = \sum_\alpha c_\alpha F_\alpha = \sum_\alpha c_\alpha F_{S(\alpha),n}$ then $K_{\overline{P}} = \sum_\alpha c_\alpha F_{\overline{S(\alpha)},n}$ where $\overline{S}$ stands for the complementary set of the set S in $[n-1]$.

For example, the dual edge-decorated poset of Figure 5 has its P-partition enumerator equal to $F_{121} + F_{112}$.

The following result appears in [6, 9, 14] and is crucial in our context, and more generally in these questions about the distinguishability of combinatorial families by the P-partition enumerator.

Theorem 2.4 ([6, 9])**.** *QSym is a unique factorization domain.*

2.2 Free quasisymmetric functions

We give a short survey of basic definitions and properties about free quasisymmetric functions that we will use.

The Hopf algebra **FQSym** of Malvenuto-Reutenauer is also called the Hopf algebra of free quasi-symmetric functions ([2, 14]). The algebra **FQSym** is the vector space generated by the elements $(\mathbb{F}_u)_{u\in\mathfrak{S}}$, where $\mathfrak{S}$ is the disjoint union of the symmetric groups S_n ($n \in \mathbb{N}$). Let us define the *shifted shuffle* $u \,\overline{⧢}\, v$ of two permutations u and v as the set of all the permutations obtained by shuffling the letters of u together with the letters of v to which are added the size of u. For example:

$12 \shuffle 21 = \{1243, 1423, 1432, 4123, 4132, 4312\}$. For any word u consisting of different integer letters, we define its standardisation $\mathrm{st}(u)$ as the (unique) permutation of the same size whose letters are in the same relative order. For example: $\mathrm{st}(4527) = 2314$. With these definitions, the product and coproduct in **FQSym** are given in the following way: for all $u \in S_n$, $v \in S_m$, by putting $u = (u_1 \dots u_n)$,

$$\begin{aligned} \Delta(\mathbb{F}_u) &= \sum_{i=0}^{n} \mathbb{F}_{\mathrm{st}(u_1 \dots u_i)} \otimes \mathbb{F}_{\mathrm{st}(u_{i+1} \dots u_n)}, \\ \mathbb{F}_u \cdot \mathbb{F}_v &= \sum_{w \in u \,\overline{\shuffle}\, v} \mathbb{F}_w. \end{aligned}$$

Its unit is $1 = \mathbb{F}_\emptyset$, where $\emptyset$ is the unique element of S_0. Moreover, **FQSym** is a $\mathbb{N}$-graded Hopf algebra, by putting $|\mathbb{F}_u| = n$ if $u \in S_n$.

Example 2.5.

$$\begin{aligned} \mathbb{F}_{(1\,2)}\mathbb{F}_{(1\,2\,3)} &= \mathbb{F}_{(1\,2\,3\,4\,5)} + \mathbb{F}_{(1\,3\,2\,4\,5)} + \mathbb{F}_{(1\,3\,4\,2\,5)} + \mathbb{F}_{(1\,3\,4\,5\,2)} + \mathbb{F}_{(3\,1\,2\,4\,5)} \\ &\quad + \mathbb{F}_{(3\,1\,4\,2\,5)} + \mathbb{F}_{(3\,1\,4\,5\,2)} + \mathbb{F}_{(3\,4\,1\,2\,5)} + \mathbb{F}_{(3\,4\,1\,5\,2)} + \mathbb{F}_{(3\,4\,5\,1\,2)}, \end{aligned}$$

$$\begin{aligned} \Delta\left(\mathbb{F}_{(1\,2\,5\,4\,3)}\right) &= 1 \otimes \mathbb{F}_{(1\,2\,5\,4\,3)} + \mathbb{F}_{(1)} \otimes \mathbb{F}_{(1\,4\,3\,2)} + \mathbb{F}_{(1\,2)} \otimes \mathbb{F}_{(3\,2\,1)} \\ &\quad + \mathbb{F}_{(1\,2\,3)} \otimes \mathbb{F}_{(2\,1)} + \mathbb{F}_{(1\,2\,4\,3)} \otimes \mathbb{F}_{(1)} + \mathbb{F}_{(1\,2\,5\,4\,3)} \otimes 1. \end{aligned}$$

3 Statistics determined by the P-partition enumerator

Much is known in the case of naturally labeled posets (see a list of properties in [1]), but far less in the general case. In this section, we recall the theory of jumps initiated in [16], slightly extending a result of [11], and we give a necessary condition for two labeled posets to have the same P-partition enumerator in the special case of posets with exactly one minimal element.

3.1 Jumps

The notion of *jump* was first considered in [16]. We recall here the main definitions and results.

Definition 3.1. Let b be an element of a labeled poset P. Let us consider the number of strict edges on a saturated chain from b down to a minimal element of P. The *jump* of b is defined as the maximal such number. In a similar way the *up-jump* of b is obtained by considering saturated chains from b up to a maximal element of P.

We introduce the statistics: $J_P^{\downarrow}(i)$ denotes the number of elements of jump equal to i in P, $J_P^{\uparrow}(j)$ the number of elements of up-jump equal to j in P, and $J_P(i,j)$ the number of elements of jump equal to i and up-jump equal to j in P.

The first result was obtained by McNarama and Ward [16] (Proposition 4.2) who proved that for any labeled poset P, the value of $J_P^{\downarrow}(i)$ is determined for any i by K.

In the naturally labeled case, Liu and Weselcouch proved ([10], Lemma 3.9) that for any i and j, the value of $J_P(i,j)$ is determined by K. We extend this to the general case. Although it is quite similar to the aforementioned result, we consider it useful to state and prove this result, since it is a powerful tool to distinguish labeled posets.

Proposition 3.2. *Let P and Q be edge-decorated posets. We have, for any i and j:*

$$K_P = K_Q \textit{ implies that } J_P(i,j) = J_Q(i,j). \tag{3}$$

Proof. We shall use Corollary 5.3 in [16] which asserts that

$$K_P = K_Q \text{ implies that } K_{P_{[i]}} = K_{Q_{[i]}}$$

where $P_{[i]}$ denotes the restriction of P to elements of jump at least i. Thus we have also that $K_{P_{[(i,j)]}} = K_{Q_{[(i,j)]}}$ where $P_{[(i,j)]}$ denotes the restriction of P to elements of both jump at least i, and up-jump at least j. By a simple degree consideration, we get: $|P_{[(i,j)]}| = |Q_{[(i,j)]}|$. We conclude by observing that

$$J_P(i,j) = |P_{[(i,j)]}| - |P_{[(i+1,j)]}| - |P_{[(i,j)]}| + |P_{[(i+1,j+1)]}|.$$

$\square$

3.2 Posets with one minimal element

Now, the following lemma is very useful when studying the P-partition enumerator of edge-decorated posets.

Lemma 3.3. *Let P be an edge-decorated poset. If P has one minimal element (that we denote by v_0), the P-partition enumerator of $P \backslash v_0$ may be computed from K_P.*

Proof. We use Theorem 2.1 and consider the linear extensions $\sigma \in \mathcal{L}(P)$. Let us denote by $a_1, \ldots, a_k$ the elements of P that cover v_0 with strict edges, and $b_1, \ldots, b_l$ the elements of P that cover v_0 with weak edges. Such a σ may be of two types: it always starts with v_0, followed by either an a_i or a b_i. In the first case σ has an initial descent, thus $F_{\mathrm{des}(\sigma),n}$ has a first part at least 2. In the second case σ has an initial ascent, thus $F_{\mathrm{des}(\sigma),n}$ has a first part equal to 1. Thus we can decompose the P-partition enumerator K in two parts: $K = K_1 + K_2$ with K_1 consisting in the F_α with $\alpha_1 = 1$ and K_2 consisting in the F_α with $\alpha_1 \geq 2$. Since the linear extensions of $P \backslash v_0$ are just those of P without the initial v_0, the P-partition enumerator of $P \backslash v_0$ is $K_1' + K_2'$ where K_1' is deduced from K_1 by removing the first part (equal to 1) in any $F_\alpha \in K_1$, and K_2' is deduced from K_2 by subtracting 1 from the first part (greater or equal to 2) in any $F_\alpha \in K_2$. $\square$

4 Fair series-parallel posets

In this section, we introduce a subclass of edge-decorated posets: *fair series parallel posets*. We prove that they are distinguished by the P-partition enumerator. This result stands in the direct continuity of the article [1]. Beforehand, we give some precise statements about quasisymmetric and free quasisymmetric functions.

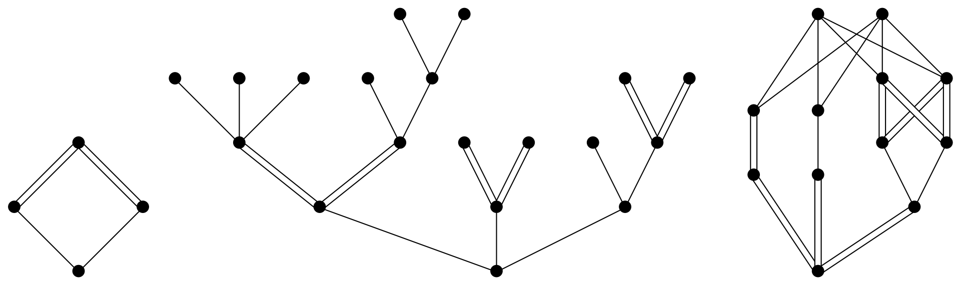


Figure 6: Examples of connected fair series-parallel posets. The second one is in particular a fair tree as of the definition of [1].

Definition 4.1. A *fair series parallel poset* is an edge decorated poset recursively defined as either:

- a single element [1],
- the poset $P \sqcup Q$ for any series-parallel posets P and Q,
- the poset $P \uparrow Q$ for any series-parallel posets P and Q, obtained from $P \sqcup Q$ by adding a weak edge (p, q) for all pairs of maximal element p of P and minimum element q of Q,
- the poset $P \Uparrow Q$ for any series-parallel posets P and Q, obtained from $P \sqcup Q$ by adding a strict edge (p, q) for all pairs of maximal element p of P and minimum element q of Q.

See Figure 6.

Fair series parallel posets are a natural generalization of fair trees of [1].

Remark 4.2. Fair series-parallel posets where all edges are weak correspond to classical series-parallel posets. They are exactly the $\boldsymbol{N}$-free posets. It is already known that they are distinguished by strict P-partition enumerators, see [11]. It remains open to characterize fair series-parallel posets as the (edge-decorated) posets avoiding a specific family of (edge-decorated) posets as subposet.

Let us introduce two operations on compositions (which are already known but we shall use notation relevant to our context):

$$(\alpha_1, \alpha_2, \ldots, \alpha_k) \uparrow (\beta_1, \beta_2, \ldots, \beta_\ell) = (\alpha_1, \alpha_2, \ldots, \alpha_k + \beta_1, \beta_2, \ldots, \beta_\ell), \tag{4}$$

and

$$(\alpha_1, \alpha_2, \ldots, \alpha_k) \Uparrow (\beta_1, \beta_2, \ldots, \beta_\ell) = (\alpha_1, \alpha_2, \ldots, \alpha_k, \beta_1, \beta_2, \ldots, \beta_\ell). \tag{5}$$

These operations give rise to two (noncommutative) products in QSym, defined on the F-basis:

$$F_\alpha \uparrow F_\beta = F_{\alpha \uparrow \beta}, \tag{6}$$

and

$$F_\alpha \Uparrow F_\beta = F_{\alpha \Uparrow \beta}. \tag{7}$$

The following proposition, proved in [1], justifies the use of the same notation as for labeled posets.

Proposition 4.3. *For any two edge-decorated posets P and Q,*

$$K_{P \uparrow Q} = K_P \uparrow K_Q,$$

and

$$K_{P \Uparrow Q} = K_P \Uparrow K_Q.$$

Without further ado, we state the expected result:

Theorem 4.4. *P-Partition enumerators distinguish fair series parallel posets.*

We emphasize the fact that there are, to our knowledge, only two theorems about injectivity of P-partition enumerators over families of posets with both weak and strict edges: Theorem 4.4 of [1], and our theorem which generalizes the latter.

It is a recurrent fact that a crucial ingredient for proving such injectivity theorems is an irreducibility result (see [5, 11]). In our case, we prove the following technical lemma.

Lemma 4.5. *The P-partition enumerator of a connected fair series-parallel poset is irreducible in $QSym$.*

Proof. Let P be a connected fair series-parallel poset. Without loss of generality, assume that $P = Q \Uparrow R$ for some fair series-parallel posets Q and R (the other case being dual, thanks to Remark 2.3). Then all linear extensions of P admit the same descent $q := |Q|$. We will show that this *global descent*, along with the homogeneity of P-partition enumerators, implies irreducibility.

For contradiction, assume that $K_P = fg$ for some non-trivial $f, g \in QSym$. Since K_P is homogeneous (say, of degree n), then so are f and g (say, of degrees n_1 and n_2 with $n_1 + n_2 = n$). Let c_α, d_β and e_δ be the coefficients of K_P, f and g, so that :

$$\sum_{\alpha \vDash n} c_\alpha F_\alpha = K_P = fg = \left(\sum_{\beta \vDash n_1} d_\beta F_\beta \right) \left(\sum_{\gamma \vDash n_2} e_\gamma F_\gamma \right). \tag{8}$$

Then, lifting everything up to **FQSym**:

$$\begin{aligned} \sum_{\sigma \in \mathfrak{S}_n} c_{\mathrm{des}(\sigma),n} \mathbb{F}_\sigma &= \left(\sum_{\nu \in \mathfrak{S}_{n_1}} d_{\mathrm{des}(\nu),n_1} \mathbb{F}_\nu \right) \left(\sum_{\tau \in \mathfrak{S}_{n_2}} e_{\mathrm{des}(\tau),n_2} \mathbb{F}_\tau \right) && (9) \\ &= \sum_{\sigma \in \mathfrak{S}_n} \left(\sum_{\sigma \in \nu \,\bar{\shuffle}\, \tau} d_{\mathrm{des}(\nu),n_1} e_{\mathrm{des}(\tau),n_2} \right) \mathbb{F}_\sigma, && (10) \end{aligned}$$

the second sum in the last right hand side term being over ν and τ. Observe that given any permutation $\sigma \in \mathfrak{S}_n$, there is only one pair $(\nu, \tau) \in \mathfrak{S}_{n_1} \times \mathfrak{S}_{n_2}$ such that $\sigma \in \nu \,\overline{\sqcup\!\sqcup}\, \tau$. Indeed, $\nu = \sigma^{[1,n_1]}$ is the permutation describing the n_1 smallest values of σ, while $\tau = \sigma^{]n_1,n]}$ is the permutation describing the n_2 biggest values of σ. Hence we get :

$$\sum_{\sigma \in \mathfrak{S}_n} c_{\mathrm{des}(\sigma),n} \mathbb{F}_\sigma = \sum_{\sigma \in \mathfrak{S}_n} d_{\mathrm{des}(\sigma^{[1,n_1]},n_1)} e_{\mathrm{des}(\sigma^{]n_1,n]},n_2)} \mathbb{F}_\sigma. \tag{11}$$

Since K_P has a global descent q, it follows that

for all $\sigma \in \mathfrak{S}_n, q \notin \mathrm{des}(\sigma)$ implies that $0 = c_{\mathrm{des}(\sigma),n} = d_{\mathrm{des}(\sigma^{[1,n_1]}),n_1} e_{\mathrm{des}(\sigma^{]n_1,n]},n_2)}$.

For all $\beta \in \mathfrak{S}_{n_1}$ and $\gamma \in \mathfrak{S}_{n_2}$ we let $\sigma_{\beta,\gamma}$ be a permutation such that:

- $q \notin \mathrm{des}(\sigma_{\beta,\gamma})$,
- $\mathrm{des}(\sigma_{\beta,\gamma}^{[1,n_1]}) = \beta$,
- $\mathrm{des}(\sigma_{\beta,\gamma}^{]n_1,n]}) = \gamma$.

It is not hard to check that such a permutation always exists. Plugging $\sigma_{\beta,\gamma}$ into Equation (11) yields:

$$\text{for all } \beta \in \mathfrak{S}_{n_1}, \gamma \in \mathfrak{S}_{n_2}, d_\beta \cdot e_\gamma = 0,$$

and it follows that either $f = 0$ or $g = 0$, which is absurd. □

Proof of Theorem 4.4. We proceed by induction on the size of the posets.

Let P and P' be fair series-parallel posets such that $K_P = K_{P'}$. We distinguish according to the shape of P. If $P = \bigsqcup_i P_i$ with every P_i a connected fair series-parallel poset, then $K_P = K_{\bigsqcup_i P_i} = \prod_i K_{P_i}$. On the other hand, $K_{P'} = K_{\bigsqcup_i P'_i} = \prod_i K_{P'_i}$ where $P' = \bigsqcup_i P'_i$. Unique factorization in $QSym$ and Lemma 4.5 conclude by induction.

If $P = Q \Uparrow R$, then as before, all linear extensions of P admit $|Q|$ as a descent, and the same goes for P' since $K_P = K_{P'}$. We claim that $P' = Q' \Uparrow R'$ for some fair series-parallel posets Q' and R' with $|Q| = |Q'| =: q$. Showing this amounts to showing that all linear extensions of P' have the same set of q first entries. Assume by contradiction that σ and ν are two linear extensions of P' such that there exist $i \in \sigma_{[1,q]} \setminus \nu_{[1,q]}$ and $j \in \nu_{[1,q]} \setminus \sigma_{[1,q]}$, and take such i (respectively j) maximizing (respectively minimizing) the position of i (respectively j) in σ. If the position of i in σ is less than q, denote by k the integer just right of i in σ. Since i is of maximal index in σ, we have $k \in \nu_{[1,q]}$, so k appears left of i in ν and right of i in σ, and i and k are incomparable in P'. As a result, exchanging i and k in σ yields another linear extension of P'. By induction, we may move i all the way to position q in σ, and the resulting permutation would still be a linear extension of P'. We proceed similarly to move j to the position $q+1$ in σ. We've just built a linear extension ℓ of P' with $\ell_q = i$ and $\ell_{q+1} = j$. Since i and j are incomparable in P' (they appear in different orders in σ and ν), $\ell \cdot (i,j)$ is also a linear extension of P'. One of these two extensions admits q as an ascent, which is absurd.

By Proposition 4.3, we have: $K_P = K_Q \Uparrow K_R$. Thus we can compute K_Q (respectively K_R) from K_P by restraining all compositions in the support of K_P (in the monomial basis) to it's first (respectively last) blocks summing up to $|Q|$ (respectively $|R|$). The same goes for $K_{P'} = K_{Q'} \Uparrow K_{R'}$, and we deduce that $K_Q = K_{Q'}$ and $K_R = K_{R'}$. By induction on the size of the graphs, these equalities imply that $Q = Q'$ and $R = R'$. Whence $P = P'$.

The case $P = Q \uparrow R$ is treated similarly. □

5 Cypress trees

Definition 5.1. A *cypress tree* is an edge-decorated poset consisting in a rooted chain with either weak or strict edges (the trunk), on which are glued leaves by weak edges.

Figure 7 shows two examples of cypress trees.

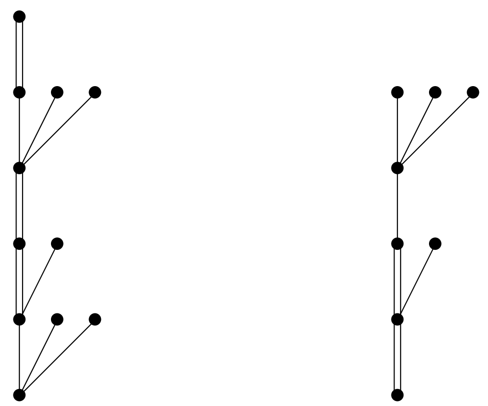


Figure 7: Cypress trees

Proposition 5.2. *Cypress trees are distinguished by the P-partition enumerator.*

The key point is the following lemma.

Lemma 5.3. *Let C be a cypress tree whose P-partition enumerator is K. It is possible to get from K the number of leaves at the root in C.*

Proof. Let us denote by a the number of leaves at the root in C.

We first deal with the case where all the edges of C are weak, which is easily tested from K, because this is equivalent to $J_C^{\downarrow}(1) = 0$. In this case we have: $a = J_C^{\downarrow}(1) - 1$.

Thus we may suppose now that C has at least one strict edge. Let us denote by $m > 0$ the number of such strict edges. We know that m may be derived from K, it is simply the maximal value of the jump statistic.

Next let us introduce the number t of vertices of the trunk (excluding the root), whose jump is zero, which coincides with the vertices of the trunk connected to the root by only weak edges. We readily observe that t is exactly the number of vertices in C with both jump equal to zero and up-jump different from zero (this positive value being nothing but m).

If the first edge of the trunk is strict (i.e. $t = 0$), then we have: $a = J^{\downarrow}_{C}(0)$. And if the first two edges of the trunk are weak (i.e. $t > 1$), then we have: $a = J^{\downarrow}_{\overline{C}}(1) - 1$.

Thus we are left with the case where the first edge of the trunk of C is weak, and the second strict (i.e. $t = 1$). We need also to define the following condition: we say that the trunk of C is a 1-trunk if it has a weak edge at the root and then only strict edges (remember we are in the case $t = 1$). We may use K to test whether the trunk of C is a 1-trunk, by testing that $J_C(i, j)$ returns 1 for $(i, j) \in \{(m, 0), (m - 1, 1), \ldots, (1, m - 1)\}$.

We now introduce $l = J_{\overline{C}}(1, 0)$. Then $l = a$ unless the trunk of C is a 1-trunk, in which case $l = a + X$, where X is the number of vertices (of the trunk) without any leaf as a descendant.

So now, if the trunk C is a 1-trunk, we consider the maximal value k such that $J^{\downarrow}_{C}(m) = 1$, $J^{\downarrow}_{C}(m-1) = 1$, $\ldots$, $J^{\downarrow}_{C}(m - k + 1) = 1$. Let us denote by v_1 and v_2 the first and second vertex of the trunk of C. We distinguish two cases:

- if $k < m$ (i.e. v_2 does not contribute to k), then $X = k$ whence $a = l - k$;
- if $k = m$ (i.e. v_2 does contribute to k), we have to decide whether v_1 has a positive number of leaves (in which case $X = k$) or not (in which case $X = k + 1$). We are done by observing that the number of leaves of v_1 is just $b = J^{\downarrow}_{\overline{C}}(2)$. □

Proof of Proposition 5.2. We will prove that we are able to reconstruct a cypress tree from its P-partition enumerator. The proof is based on an induction on the number of edges, the case of cypress with one edge being trivial.

So, let us consider a cypress tree C with $n > 1$ edges, whose P-partition enumerator is K. We know by Lemma 5.3 that we are able to get from K the number of leaves at the root in C.

We may apply Lemma 3.3 to compute the P-partition enumerator K' of the edge-decorated poset C' obtained by erasing the root of C. Then we have: $K' = F_1^a \times L$ where L is the P-partition enumerator of the cypress tree D consisting in erasing the root and its leaves in C. We may now use the induction to derive D from L. □

6 Centipedes

Definition 6.1. Let A be a word in the two letters alphabet $\{|, \|\}$. An A-*centipede* is an edge-decorated poset consisting in a rooted chain whose edges are strict or weak according to A (its *body*), on which is glued any number of up-going weak edges and down-going strict edges (its *legs*).

Figure 8 shows examples of centipedes.

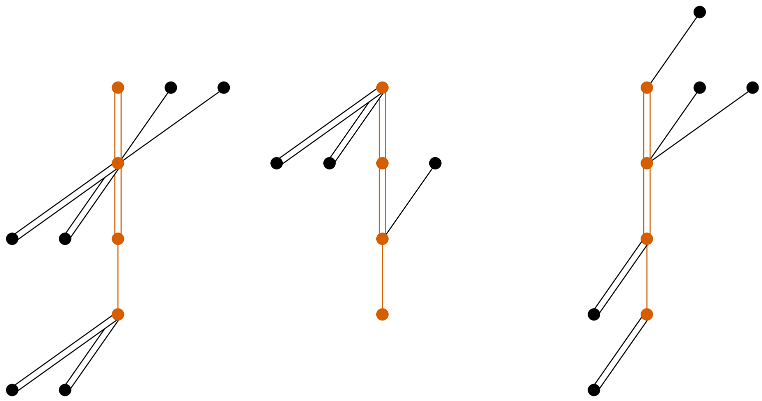


Figure 8: Three A-centipedes with $A = |, \|, \|$ and the trunk depicted in orange.

Proposition 6.2. *For any fixed $A \in \{|, \|\}^*$, A-centipedes are distinguished by the P-partition enumerator.*

Proof. We prove this statement by exhibiting the reverse bijection. Let $A \in \{|, \|\}^n$ be a word with n_1 letters $|$ and n_2 letters $\|$ and C be an A-centipede. For $i \in [n+1]$, call a_i (respectively b_i) the number of weak (respectively strict) legs attached to the i^{th} node of the body of C (starting at the bottom element).

Let k be number of $|$'s at the beginning of A, and $\ell \leq k+1$ be the maximum integer such that $b_1 = b_2 = \cdots = b_\ell$. We show that we recover ℓ from the P-partition enumerator K_C.

If $k = 0$, i.e. $A_0 = \|$, then $b_1 = J^{\uparrow}_{C}(n_2+1)$ and we know whether $\ell = 0$ or $\ell = 1$. If $k > 0$, $J^{\uparrow}_{C}(n_2+1) = b_1 + b_2 + \ldots + b_k$. If $b_1 \neq 0$, then $J^{\downarrow}_{C}(0) = \sum_{i=1}^{n+1} b_i$, and otherwise $J^{\downarrow}_{C}(0) > \sum_{i=1}^{n+1} b_i$. Since $A_1 = |$, $J^{\downarrow}_{\overline{C}}(0) = \sum_{i=1}^{n+1} b_i + 1$, and we know whether $b_1 = 0$. Proceeding by induction, suppose $b_1 = b_2 = \cdots = b_{k'}$ for some $k' < k$. Then in the same way, $J^{\downarrow}_{C}(0)$ equals (respectively is greater than) $\sum_{i=1}^{n+1} b_i + \sum_{i=1}^{k'} (a_i + 1)$. Since $J^{\downarrow}_{\overline{C}}(i) = a_i + 1$ for $1 \leq i \leq k$, we know whether $b_{k'+1}$ is null. In the end, we read the value of ℓ out of K_C through jumps.

Let us prove we can recover all a_i's, starting with a_{n+1}. If $A_n = |$, then $a_{n+1} = J^{\downarrow}_{\overline{C}}(n_1+1)$, and otherwise $a_{n+1} = J^{\downarrow}_{C}(n_2+\delta_{\ell<k+1})-1$. By induction, assume we know a_{n+1} down to a_{i+1}, and denote by m_1 (respectively m_2) the number of $|$ (respectively $\|$) in A between positions 1 and $i-1$ (included). If $A_{i-1} = |$, then $J_{\overline{C}}(m_1+1)$ is the sum of a_i, some a_j for $j > i$ and a constant. In the same way, if $A_{i-1} = \|$, then $J^{\downarrow}_{C}(m_2+\delta_{\ell<k+1})$ is the sum of a_i, some a_j for $j > i$ and a constant. These expressions only depend on ℓ, k, previous a_j's and the form of A, and hence can be read in the P-partition enumerator.

By defining dually $\overline{k}$ as the number of $\|$'s at the end of A and $\overline{\ell}$ as the number of nodes on top of the body of C with no weak leg, and performing the same kind of computations, one recovers the values of the b_i's. One can alternatively invoke a duality argument by considering the poset $\overline{C}^{\updownarrow}$ whose P-partition enumerator contains the same information as K_C. $\square$

Remark 6.3. It is not very difficult to adapt the proof to show that the P-partition enumerator distinguishes an A-centipede C from an A' centipede C' when A is a prefix of A' and both centipedes C and C' have a strict leg attached to their bottom element and a weak edge attached to their top element. We have no evidence that this leg condition is necessary, but it allows the proof to work easily.

References

[1] J.-C. Aval, K. Djenabou and P. R. McNamara, Quasisymmetric functions distinguishing trees, *Alg. Combin.* **6** (2023), 595–614.

[2] G. Duchamp, F. Hivert and J.-Y. Thibon, Noncommutative symmetric functions VI: Free quasi-symmetric functions and related algebras, *Int. J. Alg. Comp.* **12** (2002), 671–717.

[3] B. Ellzey, Chromatic quasisymmetric functions of directed graphs, *Sém. Lothar. Combin.* **78B** (2017), Art. 74.12.

[4] I. M. Gessel, Multipartite P-partitions and inner products of skew Schur functions, *Combinatorics and Algebra*, Boulder, Colo. (1983), *Contemp. Math.* **34** (1984), 289–301.

[5] T. Hasebe and S. Tsujie, Order quasisymmetric functions distinguish rooted trees, *J. Alg. Combin.* **46** (2017), 499–515.

[6] M. Hazewinkel, The algebra of quasi-symmetric functions is free over the integers, *Adv. Math.* **164** (2001), 283–300.

[7] S. Heil and C. Ji, On an algorithm for comparing the chromatic symmetric functions of trees, *Australas. J. Combin.* **75** (2019), 210–222.

[8] J. Huryn and S. Chmutov, A few more trees the chromatic symmetric function can distinguish, *Involve, J. of Math.* **13** (1) (2020), 109–116.

[9] T. Lam and P. Pylyavskyy, P-partition products and fundamental quasi-symmetric function positivity, *Adv. Appl. Math.* **40** (2008), 271–294.

[10] R. I. Liu and M. Weselcouch, P-partition generating function equivalence of naturally labeled posets, *J. Combin. Theory Ser. A* **170** (2020), 105136.

[11] R. I. Liu and M. Weselcouch, P-partitions and quasisymmetric power sums, *Int. Math. Res. Not. IMRN*, **18** (2021), 13975–14015.

[12] M. Loebl and J.-S. Sereni, Isomorphism of weighted trees and Stanley's isomorphism conjecture for caterpillars, *Ann. Inst. Henri Poincaré D* **6** (2019), 357–384.

[13] N. A. Loehr and G. S. Warrington, Combinatorics of multivariate chromatic polynomials for rooted graphs, `arXiv:2206.05392`.

[14] C. Malvenuto and C. Reutenauer, Duality between quasi-symmetric functions and the Solomon descent algebra, J. Algebra **177** (1995), 967–982.

[15] J. L. Martin, M. Morin and J. D. Wagner, On distinguishing trees by their chromatic symmetric functions, *J. Combin. Theory Ser. A* **115** (2008), 237–253.

[16] P. R. McNamara and R. E. Ward, Equality of P-partition generating functions, *Ann. Comb.* **18** (2014), 489–514.

[17] J. Shareshian and M. L. Wachs, Chromatic quasisymmetric functions, *Adv. Math.* **295** (2016), 497–551.

[18] R. P. Stanley, Ordered structures and partitions, Mem. Amer. Math. Soc. No. 119, 1972.

[19] R. P. Stanley, A symmetric function generalization of the chromatic polynomial of a graph, *Adv. Math.* **111** (1995), 166–194.

[20] J. Zhou, Reconstructing rooted trees from their strict order quasisymmetric functions, `arXiv:2008.00424`.

(Received 29 Jan 2024; revised 28 Sep 2024)